MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

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CONTENTS.

Solved Questions.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

6782. (By R. A. ROBERTS, M.A.)—In the motion of a rigid body about a fixed point under the action of no forces, show that the plane, containing two positions of the invariable line in the body which are separated by a constant interval of time, touches a cone concyclic with the invariable cone.

Solution by Professor TOWNSEND, F.R.S.

Assuming that a great circle of a sphere, which cuts off a constant spherical area from any fixed conic on the sphere envelopes a concyclic conic on the sphere, in order to prove the above pretty property, it is only necessary to show that, for every two positions OP_1 and OP_2 of the invariable line in the body, its two momentary changes of position P_1OQ_1 and P_2OQ_2 are inversely as the sines of the angles which their planes make with the plane of connection P_1OP_2 of the positions. This may be readily done as follows.

It was shown by MACCULLAGH, at his lectures on the rotation of a rigid body round a fixed point under the circumstances supposed in the Question (see his *Collected Works*, p. 338), that, in any position OP of the invariable line in the body, its momentary change of position POQ varies as the quantity
$$\left\{ \left(\frac{a^2-r^2}{a^2}\right)^2 x^2 + \left(\frac{b^2-r^2}{b^2}\right)^2 y^2 + \left(\frac{c^2-r^2}{c^2}\right)^2 z^2 \right\}^{\frac{1}{2}},$$
where a, b, c are the semi-axes of the ellipsoid of inertia of the body with respect to the centre of rotation O, r the constant length of the invariable radius of the motion, and x, y, z the coordinates perpendicular to the principal planes of the surface of the corresponding position P of the invariable point in the body. Hence, in the spherical triangle P_1IP_2 determined, on the sphere of radius r round centre O, by the plane P_1OP_2 connecting any two positions OP_1 and OP_2 of the invariable line, with the two tangent planes OP_1I and OP_2I to the invariable cone corresponding to the positions, it is to be shown that
$$\left\{ \left(\frac{a^2-r^2}{a^2}\right)^2 x_1^2 + \&c. \right\} \sin IP_1P_2 = \left\{ \left(\frac{a^2-r^2}{a^2}\right)^2 x_2^2 + \&c. \right\}^{\frac{1}{2}} \sin IP_2P_1,$$

or, if P be the pole on the sphere of the great circle P_1P_2, that

$$\left\{\left(\frac{a^2-r^2}{a^2}\right)^2 x_1^2+\&c.\right\}^{\frac{1}{2}}\cos IP_1P=\left\{\left(\frac{a^2-r^2}{a^2}\right)^2 x_2^2+\&c.\right\}^{\frac{1}{2}}\cos IP_2P.$$

But, the equation of the invariable cone being $\left(\frac{a^2-r^2}{a^2}\right)x^2+\&c.=0$, and those of the tangent planes to it along OP_1 and OP_2 being consequently $\left(\frac{x^2-a^2}{a^2}\right)x_1\,.\,x+\&c.=0$ and $\left(\frac{x^2-a^2}{a^2}\right)x_2\,.\,x+\&c.=0$, it is easily seen, after a few trifling reductions, that the two magnitudes to be proved equal, multiplied each by the factor $r\,.\sin P_1OP_2$, have for common value the symmetrical equivalent

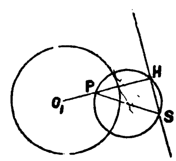
$$\left\{\left(\frac{a^2-r^2}{a^2}\right)x_1x_2+\left(\frac{b^2-r^2}{b^2}\right)y_1y_2+\left(\frac{c^2-r^2}{c^2}\right)z_1z_2\right\};$$

and therefore &c., as regards the property in question.

5905. (By Professor Cochez.)—Quatre circonférences étant données, trouver sur l'une d'elles un point tel que les polaires de ce point par rapport aux trois autres circonférences le coupent au même point.

Solution by J. L. McKenzie, B.A.; E. Rutter; *and others.*

Suppose O_1, O_2, O_3 to be the centres of the three circles; P the required point; S the point of intersection of the three polars of P. Let HS be the polar of P with regard to the circle O_1; then O_1P is perpendicular to HS; and $O_1H\,.\,O_1P=r_1^2$, where r_1 = radius of the circle O_1. Therefore the circle PHS (on PS as diameter) cuts the circle O_1 orthogonally. Similarly the circle on PS as diameter cuts the circles O_2 and O_3 orthogonally. Hence the construction; draw the circle cutting orthogonally the three circles O_1, O_2, O_3; either of the points in which it cuts the fourth given circle may be taken as the required point.



[The problem is impossible in the following cases:—

(1) If, of the three circles O_1, O_2, O_3, each cuts the other two, seeing that, then, no circle can be drawn cutting the three orthogonally; (2) if the circle orthogonal to $O_1O_2O_3$ does not meet the fourth circle.]

6905. (By Professor Hudson, M.A.)—A heavy pulley, of weight R, hangs by a string over a fixed pulley, the other end of the string being attached to a weight W; over the pulley R hangs a string with unequal weights P, Q at its ends; find the accelerations of P, Q, W; and, as an example, let P = 5 lbs., Q = 4 lbs., R = 2 lbs., W = 12 lbs.

Solution by G. HEPPEL, M.A.; C. BICKERDIKE; *and others.*

Let the accelerations of P, Q, R, W, be respectively pg, qg, rg, wg, all being estimated downwards; then, equating the tensions of the two parts of lower and upper strings, and expressing the condition that the strings may remain stretched, we have

$$P(1-p) = Q(1-q), \quad W(1-w) = R(1-r) + 2P(1-p),$$
$$w + r = 0, \quad p - r + q - r = 0,$$

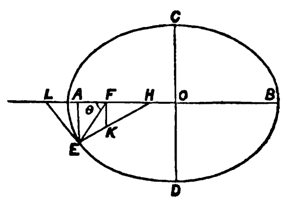
4 equations to determine p, q, w, r. Putting $W + R = U$, $W - R = V$, $P + Q = S$, $P - Q = D$, the results are

$$w = \frac{VS - 4PQ}{US + 4PQ}, \quad p = -\frac{VS - 4PQ - 2DW}{US + 4PQ}, \quad q = -\frac{VS - 4PQ + 2DW}{US + 4PQ}.$$

Substituting the numbers given in the question, we get the acceleration of W $= \frac{5}{103}g$ downwards, of P $\frac{7}{103}g$ downwards, and of Q $\frac{17}{103}g$ upwards.

7048. (By Dr. CURTIS.)—An ellipse is placed with its major axis horizontal; prove that the straight line of quickest descent from either focus to the curve (or conversely) is given by the equation $r^2 - 3ar + 2b^2 = 0$, and that *only one* of the roots of this equation for r, the focal radius vector, is admissible. [See Question 6970, *Reprint*, Vol. xxxvii., p. 62.]

Solution by J. YOUNG, M.A.; ROBERT RAWSON; *and others.*



Let the ellipse be represented in the figure, O being its centre and F the focus with which we are concerned. The radius vector FE, which solves the problem, must be such that a circle can be described which shall touch the major axis AB at F and the ellipsoid at E; the point E must therefore be such that, drawing the normal EH cutting in K the perpendicular to AB at F, FK shall $=$ EK, and therefore, drawing the tangent EL, EL $=$ LF. Denoting, then, the angle AFE by θ, we have $2\text{LF}\cos\theta = r$; or, if x be the abscissa of E referred to the axes of the ellipse,

$$2\left(\frac{a^2}{x} - ae\right)\cos\theta = r, \quad \text{or} \quad 2a(a - ex)\cos\theta = rx;$$

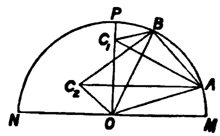
but $r = a - ex$, therefore $2a\cos\theta = x$, or $2ae\cos\theta = ex = a - r$;

but, by the equation of the ellipse, $r = \dfrac{\frac{b^2}{a}}{1 + e\cos\theta}$, $\therefore 2a\left(\dfrac{b^2}{ar} - 1\right) = a - r$,

or $r^2 - 3ar + 2b^2 = 0$; hence $r = \dfrac{3a}{2} \pm \dfrac{a}{2}\left(9 - 8\dfrac{b^2}{a^2}\right)^{\frac{1}{2}}$; but, as $9 - 8\dfrac{b^2}{a^2} > 1$, and $r < 2a$, the upper sign is excluded.

6129. (By Professor MATZ, M.A.)—Two points are taken at random in the arc of a given circular quadrant of radius I; and a third point is taken at random, (1) in one of the rectilinear sides of the quadrant, and (2) in the surface of the *adjacent* quadrant; and these are joined, in their respective order, by straight lines; prove that the mean areas of the triangles thus formed are $\Delta_1 = \frac{r^2}{\pi^2}(3\pi-8)$, $\Delta_2 = \frac{r^2}{\pi^2}(2\pi-4)$.

Solution by Professor SEITZ, M.A.; D. EDWARDES; *and others.*



Let MOP and NOP be the two quadrants; A, B two random points in the arc MP; C_1, C_2 random points in OP and the surface NOP, respectively; $OM = r$, $OC_1 = x$, $OC_2 = y$, $\angle AOP = \theta$, $\angle BOP = \phi$, $\angle POC_2 = \psi$, area $ABC_1 = u_1$, and area $ABC_2 = u_2$; then

$$u_1 = \tfrac{1}{2}r^2 \sin(\theta-\phi) + \tfrac{1}{2}rx\sin\phi - \tfrac{1}{2}rx\sin\theta,$$

$$u_2 = \tfrac{1}{2}r^2 \sin(\theta-\phi) + \tfrac{1}{2}ry\sin(\phi+\psi) - \tfrac{1}{2}ry\sin(\theta+\psi); \text{ hence}$$

$$\Delta_1 = \int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{r} u_1\, r d\theta\, r d\phi\, dx \div \int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{r} r d\theta\, r d\phi\, dx$$

$$= \frac{2r^2}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left\{2\sin(\theta-\phi)+\sin\phi-\sin\theta\right\} d\theta\, d\phi$$

$$= \frac{2r^2}{\pi^2}\int_0^{\frac{1}{2}\pi}(3-3\cos\theta-\theta\sin\theta)\, d\theta = \frac{r^2}{\pi^2}(3\pi-8);$$

$$\Delta_2 = \int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{\frac{1}{2}\pi}\int_0^{r} u_2\, r d\theta\, r d\phi\, d\psi\, y\, dy \div \int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{\frac{1}{2}\pi}\int_0^{r} r d\theta\, r d\phi\, d\psi\, y\, dy$$

$$= \frac{8r^2}{3\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{\frac{1}{2}\pi}\left\{3\sin(\theta-\phi)+2\sin(\phi+\psi)-2\sin(\theta+\psi)\right\} d\theta\, d\phi\, d\psi$$

$$= \frac{4r^2}{3\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left\{3\pi\sin(\theta-\phi)-4\sin\theta-4\cos\theta+4\sin\phi+4\cos\phi\right\} d\theta\, d\phi$$

$$= \frac{4r^2}{3\pi^3}\int_0^{\frac{1}{2}\pi}\left\{3\pi-3\pi\cos\theta-4\theta\sin\theta-4\theta\cos\theta+4+4\sin\theta-4\cos\theta\right\} d\theta$$

$$= \frac{r^2}{\pi^2}(2\pi-4).$$

6805. (By A. MCINTOSH, M.A.)—In Question 6784, prove that (1) the line joining the points of contact of the conic in (14) with the nodal tangents, passes through the points of inflexion of the cubic; (2) if two imaginary tangents be drawn from the point R (*i.e.*, the third point in which TT′ meets the curve), prove that the chord joining the imaginary points of contact will pass through the primary point P; (3) the point in which PN meets TT′ lies on another nodal cubic which has the same

nodal tangents and the same conic as in (14, 6487); (4) if two real tangents be drawn from the point of intersection of PN and TT′ to this latter cubic, the line joining the points of contact is the same as the line mentioned in (2), and therefore passes through P; (5) a line drawn through a fixed point on a nodal cubic intersects the curve again in two points; if two pairs of tangents be drawn from these points to the curve, the chords joining their points of contact intersect on a fixed line passing through the node; (6) the point in which this fixed line meets the curve, and the line joining the first fixed points to the node, bear the same relation to one another; (7) in a cuspidal cubic, if the line joining any two points on the curve subtends a right angle at the cusp, it touches a conic which passes through the cusp, and cuts the cubic there at right angles.

Solution by the PROPOSER; G. F. WALKER, M.A.; *and others.*

Equation of Folium being $x^3+y^3=3axy$, any point on the curve may be given by the equations $x=\dfrac{3a\mu}{1+\mu^3}$ $y=\dfrac{3a\mu^2}{1+\mu^3}$,
μ being the tangent of the angle that the line joining the point to the node makes with one of the nodal tangents, if the nodal tangents are at right angles.

Let a tangent touch at (xy) and pass through $(x'y')$, the latter being a point on the curve. Then

$$(x^2-ay)(x-x')+(y^2-ax)(y-y')=0.$$

This reduces to $\qquad (x^2-ay)\,x'+(y^2-ax)\,y'=axy,$

by the equation of the curve.

Substitute in this equation the values given above for xy, and for $x'y'$ substitute respectively $\dfrac{3a\mu'}{1+\mu'^3}$ and $\dfrac{3a\mu'^2}{1+\mu'^3}$. It will be found that the equation may be reduced to $(\mu-\mu')^2(\mu^2\mu'+1)=0$. This shows that $\mu^2\mu'=-1$. If μ' be positive, *i.e.*, if the point with which we start lie on the loop, the values of μ are imaginary, therefore two real tangents cannot be drawn from any point on the loop.

The two values of μ are equal and opposite in sign, therefore NT + NT′ make equal angles with the nodal tangents in the Folium. The coordinates of the middle point of TT′ are

$$\tfrac{1}{2}\left(\frac{3a\mu}{1+\mu^3}-\frac{3a\mu}{1-\mu^3}\right) \text{ and } \tfrac{1}{2}\left(\frac{3a\mu^2}{1+\mu^3}+\frac{3a\mu^2}{1-\mu^3}\right).$$

These expressions become $-\dfrac{3a\mu^4}{1-\mu^6}$ and $\dfrac{3a\mu^2}{1-\mu^6}$ respectively. But the coordinates of P are $\dfrac{3a\mu'}{1+\mu'^3}$ and $\dfrac{3a\mu'^2}{1+\mu'^3}$, *i.e.*, $\dfrac{3a\mu^4}{1-\mu^6}$ and $-\dfrac{3a\mu^2}{1-\mu^6}$.

These are equal and opposite in sign to the coordinates of M, the middle point of TT′; therefore PM passes through N, and is bisected in N.

The equation of the line TT′ is easily found. It is

$$\frac{x-\dfrac{3a\mu}{1+\mu^3}}{\dfrac{3a\mu}{1+\mu^3}+\dfrac{3a\mu}{1-\mu^3}}=\frac{y-\dfrac{3a\mu^2}{1+\mu^3}}{\dfrac{3a\mu^2}{1+\mu^3}-\dfrac{3a\mu^2}{1-\mu^3}},$$

therefore $$\frac{x(1+\mu^3)-3a\mu}{6a\mu} = \frac{y(1+\mu^3)-3a\mu^2}{-6a\mu^5}.$$

Hence $(x\mu^4+y)(1+\mu^3)-3a\mu^2(1+\mu^3)=0$, or $x\mu^4+y-3a\mu^2=0$.

If a tangent be drawn at the point P, the μ of the point in which it meets the curve again is $-\frac{1}{\mu'^2}$; but $\mu' = -\frac{1}{\mu^2}$, therefore the μ of the point in which it meets the curve again is $-\mu^4$. Hence the equation of the line joining N to this point is $y+x\mu^4=0$. This line is consequently parallel to TT′, and therefore the line PQ in the question is bisected by the curve.

It is evident, from the equation of the line TT′, that it envelopes the rectangular hyperbola, $4xy=9a^2$. The hyperbola touches the curve in the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, and this point is the vertex. It will be easily seen that the chords joining real points of contact touch one branch of the hyperbola, and the chords joining imaginary points of contact, the other.

If the tangent PQ cut the curve in S, it follows from the general theory of cubics that RS is also a tangent to the cubic at the point R. The chord RS, therefore, bears the same relation to the cubic as TT′, and therefore also touches the rectangular hyperbola mentioned above. The lines NT and NT′ make equal angles with a nodal tangent and for the same reason, the lines NM and NR make equal angles with it, therefore $\angle$TNM = $\angle$T′NR.

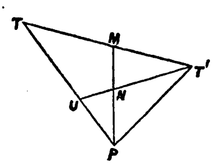
The middle point M of TT′ lies on an identical Folium, because MN is equal to PN and is in an opposite direction. It is thus easy to see that the equation of this Folium is $x^3+y^3=-3axy$, and that its loop lies in the opposite quadrant.

The coordinates of the point M are $-\frac{3a\mu'}{1+\mu'^3}$ and $-\frac{3a\mu'^2}{1+\mu'^3}$: the equation of the chord joining the points of contact of tangents from this point to the curve $x^3+y^3=-3axy$ is therefore

$$\frac{1}{\mu'^2}x+y-\frac{3a}{\mu'}=0, \quad i.e., \quad x+\mu'^2y-3a\mu'=0.$$

This line passes through the point $\left(\frac{3a\mu'}{1+\mu'^3}, \frac{3a\mu'^2}{1+\mu'^3}\right)$, as may be verified by substitution, and it also touches one branch of the rectangular hyperbola mentioned before. It also passes through the imaginary points of contact of tangents from the point R to the original curve. This is very much like the theory of poles and polars with respect to curves of the second degree.

If T′N be produced on to meet the tangent PT, we see, by the theory of transversals, that



$$\text{TT}'.\text{MN}.\text{PU} = \text{MT}'.\text{PN}.\text{TU};$$

but TT′ = 2MT′ and MN = PN,

therefore TU = 2PU;

similarly TM . T′N . TP = T′M . NU . PU;

but TM = T′M and TP = 3PU

therefore T′N = 3NU.

Therefore U, the point of trisection of PT most remote from the point of contact, lies on a Folium, the linear dimensions of which are one-third those of the original. The centroid of the triangle PTT′ evidently lies on this Folium. It is a point of trisection of the tangent at R to the curve. This hardly requires a proof.

Taking the general nodal cubic $ax^3+3bx^2y+3cxy^2+dy^3=3exy$, and proceeding in the same manner, we may prove that $\mu^2\mu'=-\dfrac{a}{d}$. However, it will be simpler to find the μ's for the points in which any line $y=mx+n$ cuts this curve. The μ's will be the three ratios of y to x given by the cubic equation $(ax^3+3bx^2y+3cxy^2+dy^3)\,n=3exy\,(y-mx)$. Hence $\mu\mu'\mu''=-\dfrac{an}{dn}=-\dfrac{a}{d}$. If the line be a tangent, two of the μ's will be equal, therefore $\mu^2\mu'=-\dfrac{a}{d}$; $\mu=\pm\sqrt{-\dfrac{a}{d\mu'}}$, $\therefore$ NT and NT′ form a harmonic pencil with the nodal tangents. A point on the curve is now given by
$$x=\frac{3e\mu}{a+3b\mu+3c\mu^2+d\mu^3},\quad y=\frac{3e\mu^2}{a+3b\mu+3c\mu^2+d\mu^3}.$$
Let this be the point T. The coordinates of the point T′ are
$$\frac{-3e\mu}{a-3b\mu+3c\mu^2-d\mu^3}\text{ and }\frac{3e\mu^2}{a-3b\mu+3c\mu^2-d\mu^3}.$$
The equation of the chord TT′ can now be worked out as before. It will be found to be divisible by $a+3b\mu+3c\mu^2+d\mu^3$ finally, just as the other was divisible by $1+\mu^3$. It becomes $\mu^4dx+(3bx+3cy-e)\,\mu^2+ay=0$, and therefore touches the conic $4adxy=9\,(bx+cy-e)^2$, a conic which evidently touches the nodal tangents, the chord of contact being
$$bx+cy-e=0.$$

6744. (By Dr. Macfarlane, F.R.S.E.)—Deduce the necessary consequences of the law that a man cannot be the husband of his wife's sister.

Solution by the Proposer.

Let A denote any subject of a nation in which the above law is established; then mA will denote any male subject, and $mc^{-1}cfcc^{-1}fc^{-1}cm$A will denote any husband of any sister of any wife of the man A. Hence $\Sigma\, mc^{-1}cfcc^{-1}fc^{-1}cm$A will denote all the husbands of all the sisters of all the wives of the man A. The law referred to provides that A and any member of the above class cannot be identical, which is expressed by
$$\Sigma\text{A}\,.\,mc^{-1}cfcc^{-1}fc^{-1}cm\text{A}=0\text{........................(1)},$$
where the left-hand side denotes all the members of the compound class formed by taking simultaneously the two classes A and $\Sigma\, mc^{-1}cfcc^{-1}fc^{-1}cm$A. The rule for transforming such a universal equation is as follows:—Suppose all the relationship symbols brought over to one factor of the com-

pound class in accordance with the rule given in solution to Question 6677 (*Reprint*, Vol. XXXV, p. 110); then, removing a symbol from the front gives one derived equation, and removing a symbol from the end gives another derived equation; transform each of these derived equations in a similar manner, then each of their four resultants, and so on, until all the relationship symbols have been brought to the other side. The total of these derived equations forms the total of the transformations of the given universal equation. The first derived equation in the above case is

$$\Sigma cmA \,.\, cfcc^{-1}fc^{-1}cmA = 0 \quad\text{.......................(2)};$$

that is, a child of a man cannot be the child of a sister of a wife of the man. The second derived equation is

$$\Sigma mc^{-1}A \,.\, mc^{-1}cfcc^{-1}fc^{-1}A = 0 \quad\text{.....................(3)};$$

that is, the father of a person cannot be the husband of a sister of the mother of the person. By applying the process to (2) and (3), and so on, other 19 different forms are obtained, which respectively express that—

A wife of a man cannot be the sister of a wife of the man.
A child of the father of a person* ,, the child of a sister of the mother of the person.
A husband of a woman ,, the husband of a sister of the woman.
A parent of a wife of a man ,, the parent of another wife of the man.
A wife of the father of a person ,, the sister of the mother of the person.
A child of a husband of a woman ,, the child of a sister of the woman.
A husband of a daughter of a person ,, the husband of another daughter of the person.
A parent of a step-mother of a person ,, the parent of the mother of the person.
A wife of a husband of a woman ,, the sister of the woman.
A child of a husband of a daughter of a person ,, the child of another daughter of the person.
A sister of the wife of the father of a person ,, the mother of the person.
A parent of another wife of a husband of a woman ,, the parent of the woman.
A second wife of a husband of a daughter of a person ,, the daughter of the person.
A child of a husband of a sister of a woman ,, the child of the woman.
A child of a sister of a wife of the father of a person ,, the person.
A sister of a wife of a husband of a woman ,, the woman.
A parent of another wife of a husband of a daughter of a person ,, the person.
A wife of a husband of a sister of a woman ,, the woman.
A child of a husband of a sister of the mother of a person ,, the person.

6913, 6940. (By Rev. A. J. C. Allen, M.A., and J. Young, M.A.)—If K be the orthocentre of a triangle, P any point on the circumscribing circle, and if PK cut the pedal line of P in Q; prove that (1) PK is bisected in Q, (2) the locus of Q is the nine-point circle.

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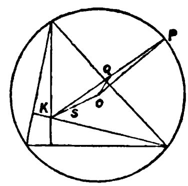
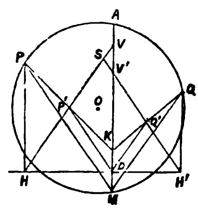
* The symbol ,, is an ellipse for *cannot lawfully be*.

6916. (By R. Tucker, M.A.)—ABC is a triangle, and PQR is an equilateral triangle in the circle ABC; prove that (1) the pedal lines, corresponding to the points P, Q, R, form another equilateral triangle; (2) the pedal lines, corresponding to points at equal arcual distances from any angle (A), intersect on a straight line.

Solution by Charlotte A. Scott; Sarah Marks; T. Woodcock, B.A.; *and others.*



(6913, 6940). Since (*Reprint*, XXXIV, 25) KP is bisected in Q, if we join K to O, the centre of the circumscribing circle, and bisect KO in S, we have $$SQ = \tfrac{1}{2}OP;$$
hence the locus of Q is the nine-point circle.



(6916). 1. The angle PMA = HVM,
and the angle QMA = H'V'M;
thus, if HP' and H'Q' intersect in S, so that
$$\angle HSH' = HVM + H'VM,$$
we have $\angle HSH' = PMA + QMA = PMQ = PRQ$;
hence the pedal lines form a triangle similar to the triangle PQR, that is, an equilateral triangle.

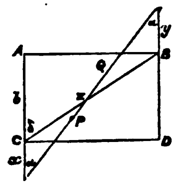
2. P' is the middle point of KP, and D of KM, hence P'D is parallel to PM, that is, P'D and P'V are equally inclined to VD. Similarly; Q'D and Q'V' are equally inclined to VD. If AP = AO, $\angle AMP = AMQ$, and therefore $\angle P'DV = Q'DV' = Q'V'D$, therefore DP'SQ' is a parallelogram, therefore DS and P'Q' bisect each other. But P'Q' is fixed in direction, and is a chord of the nine-point circle, therefore the middle point of P'Q' lies on a fixed diameter of the nine-point circle, that is, the middle point of DS lies on a fixed line, and therefore, D being a fixed point, the locus of S is a straight line.

[Any point P on the circumscribed circle will be the focus of a parabola touching the sides of the triangle; hence the feet of the perpendiculars from P upon the sides of the triangle lie on a straight line, viz., the tangent at the vertex of this parabola; whence we have a proof of the existence of the pedal line. The directrix of this parabola passes through the orthocentre, and is parallel to the pedal line of P. Therefore the straight line from P to the orthocentre is bisected by the pedal line, which proves the property in Question 6940. The centroid and the orthocentre of a triangle are the (internal and external) centres of similitude of the nine-point and circumscribing circles, whose radii are in the ratio 1 : 2; hence, if K be the orthocentre, the mid-point of PK lies on the nine-point circle, which proves Question 6913.]

6135. (By W. S. B. Woolhouse, F.R.A.S.)—Find (1) the average area of triangles drawn on the surface of a given rectangle, and having

one of the three sides parallel to a given line; and (2) the directions of the given line when the average area is a maximum and a minimum.

Solution by the PROPOSER.



Parallels to the given line will meet only one of the pairs of opposite sides of the rectangle. Let AB and CD be these sides, and the notation as in the diagram; then we have two cases to consider.

1. When PQ crosses AB and CD (see Diagram 1). PQ being the side of the triangle, let p denote the perpendicular from a variable point R as vertex, and a, b the sides AB, AC; then we have

$$\Sigma p = \frac{(x+b)^2 \tan\alpha}{2} \cdot \frac{(x+b)\sin\alpha}{3} - \frac{x^2\tan\alpha}{2} \cdot \frac{x\sin\alpha}{3} + \frac{(b+y)^2\tan\alpha}{2} \cdot \frac{(b+y)\sin\alpha}{3} - \frac{y^2\tan\alpha}{3} \cdot \frac{y\sin\alpha}{3}$$

$$= \frac{\sin^2\alpha}{6\cos\alpha}\left\{(b+x)^3 - x^3 + (b+y)^3 - y^3\right\}.$$

Also, when the points P, Q vary in the line,

$$\Sigma\,\mathrm{PQ} = \int z\,dz\left(\frac{b}{\cos\alpha} - z\right) = \frac{b}{\cos\alpha} \cdot \frac{z^2}{2} - \frac{z^3}{3} = \frac{b^3}{6\cos^3\alpha}.$$

Therefore, as regards the line PQ,

$$\Sigma\Delta' = \tfrac{1}{2}\,\Sigma p \,.\, \Sigma\mathrm{PQ} = \frac{b^3\sin^2\alpha}{72\cos^4\alpha}\left\{(b+x)^3 - x^3 + (b+y)^3 - y^3\right\}.$$

To involve variable positions of the line, multiply by $dx \sin\alpha = -dy\sin\alpha$ and integrate, and we obtain

$$\frac{b^3\sin^3\alpha}{288\cos^4\alpha}\left\{(b+x)^4 - x^4 - (b+y)^4 + y^4\right\}.$$

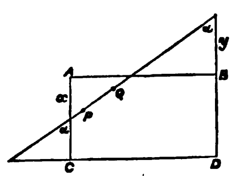
This is to be taken between the limits

$$\begin{pmatrix} x = 0 \\ y = \dfrac{a}{\tan\alpha} - b \end{pmatrix} \text{ and } \begin{pmatrix} x = \dfrac{a}{\tan\alpha} - b \\ y = 0 \end{pmatrix}$$

and the result is

$$\frac{b^4}{72\cos\alpha}\left(2a^3 - 3a^2 b\tan\alpha + 2ab^2\tan^2\alpha - b^3\tan^3\alpha\right) \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

2. When the line PQ crosses AB and AC.



$$\Sigma p = \frac{\sin^2\alpha}{6\cos\alpha}\left\{x^3 + (b+y)^3 - (b-x)^3 - y^3\right\},$$

$$\Sigma\mathrm{PQ} = \frac{x^3}{6\cos^3\alpha},$$

$$\Sigma\Delta' = \frac{x^3\sin^2\alpha}{72\cos^4\alpha}\left\{x^3 + (b+y)^3 - (b-x)^3 - y^3\right\}$$

$$= \frac{x^3\sin^2\alpha}{72\cos^4\alpha}\left\{x^3 + \left(b + \frac{a}{\tan\alpha} - x\right)^3 - (b-x)^3 - \left(\frac{a}{\tan\alpha} - x\right)^3\right\}$$

$$= \frac{x^3\sin^2\alpha}{72\cos^4\alpha}\left\{\frac{3ab}{\tan\alpha}\left(b + \frac{a}{\tan\alpha}\right) - \frac{6ab}{\tan\alpha}x + 2x^3\right\},$$

which multiplied by $dx \sin\alpha$ and integrated between the limits $x = b \ldots 0$ we obtain
$$\frac{b^4}{72\cos\alpha}\left(\tfrac{3}{4}a^2b\tan\alpha - \tfrac{9}{20}ab^2\tan^2\alpha + \tfrac{2}{7}b^3\tan^3\alpha\right) \;\ldots\ldots\ldots\ldots(2).$$
The same values again recur when the line PQ crosses CD and BD. Hence taking the former result (1), and adding the double of (2), the total sum of all the triangles is found to be
$$\frac{b^4}{72\cos\alpha}\left(2a^3 - \tfrac{3}{2}a^2b\tan\alpha + \tfrac{11}{10}ab^2\tan^2\alpha - \tfrac{3}{7}b^3\tan^3\alpha\right)\ldots\ldots\ldots\ldots(3).$$
To ascertain the number of triangles, the positions of P, Q are in the first case
$$\int dx \sin\alpha \,.\, \frac{b^2}{2\cos^2\alpha} = \frac{b^2}{2\cos\alpha}\,(a - b\tan\alpha),$$
and in the second
$$\int dx \sin\alpha \,.\, \frac{x^2}{2\cos^2\alpha} = \frac{b^2}{2\cos\alpha}\,\frac{b\tan\alpha}{3},$$
so that the total positions of P, Q are $\dfrac{b^2}{2\cos\alpha}\left(a - \tfrac{2}{3}b\tan\alpha\right)$. And, as each of these has ab positions of the third point R, the total number of triangles is $\dfrac{ab^3}{2\cos\alpha}\left(a - \tfrac{2}{3}b\tan\alpha\right)$. Divide (3) by this, and the average triangle is
$$u = \frac{ab}{36}\,.\,\frac{2a^3 - \frac{3}{2}a^2b\tan\alpha + \frac{11}{10}ab^2\tan^2\alpha - \frac{3}{7}b^3\tan^3\alpha}{a^2\left(a - \frac{2}{3}b\tan\alpha\right)}.$$
To simplify, let
$$k = \frac{b\tan\alpha}{a}\left(= \frac{\tan\alpha}{\tan\delta}\right);$$
then
$$u = \frac{ab}{36}\,.\,\frac{2 - \frac{3}{2}k + \frac{11}{10}k^2 - \frac{3}{7}k^3}{1 - \frac{2}{3}k} = \frac{ab}{840}\left\{30k^2 + 13k + 144 - \frac{292}{3-k}\right\}.$$
Also
$$\frac{du}{dk} = \frac{ab}{840}\left\{60k + 13 - \frac{292}{(3-k)^2}\right\}, \qquad \frac{d^2u}{dk^2} = \frac{ab}{210}\left\{15 - \frac{146}{(3-k)^3}\right\}.$$
For maxima and minima areas, $\dfrac{du}{dk}$ put $= 0$ resolves into
$$(k-1)(60k^2 - 287k + 175) = 0.$$
The root $k = 1$ determines a maximum area, and the inferior root of the quadratic a minimum area. As k must not exceed 1, the superior root is not admissible. The greatest area $(= \frac{1}{18}ab)$ obtains when $k = 0$.

[Another solution has been given on p. 109 of Vol. 37 of the *Reprints*.]

6733. (By G. F. Walker, M.A.)—Prove that the result of eliminating the arbitrary function from $F(X_1, X_2, \ldots X_r, \ldots X_n) = 0$, where
$$X_r = S^\lambda(x_r - u),\quad S = x_1 + \ldots + x_n + u, \text{ and } x_1 \ldots x_n,$$
are independent variables, is
$$\sum_{r=1}^{r=n}\left[-\lambda(n+1)x_r + (1+\lambda)S\right]\frac{du}{dx_r} = -\lambda(n+1)u + (1+\lambda)S.$$

Solution by the PROPOSER; Prof. EVANS, M.A.; *and others.*

$F=0$ is the integral of a partial differential equation of the first order, the integrals of the auxiliary equations of which are $X_1=a_1, \dots X_n=a_n$, and therefore, from the auxiliary equations, we have $dX_1=0 \dots dX_n=0$,

therefore $$-\frac{\lambda ds}{s}=\frac{dx_1-du}{x_1-u}=\dots=\frac{dx_r-du}{x_r-du}=\dots$$

$$=\frac{ds-(x+1)\,du}{s-(x+1)\,u}=\frac{-\lambda\,(n+1)\,du}{(1+\lambda)\,s-\lambda\,(n+1)\,u}=\frac{-\lambda\,(n+1)\,dx_r}{(1+\lambda)\,s-\lambda\,(n+1)\,x_r}=\dots,$$

or $$\frac{dx_1}{-\lambda\,(n+1)\,x_1+(1+\lambda)\,s}=\dots=\frac{dx_r}{-\lambda\,(n+1)\,x_r+(1+\lambda)\,s}=\dots$$

$$\dots=\frac{du}{-\lambda\,(1+x)\,u+(1+\lambda)\,s}.$$

The differential equation therefore is

$$\sum_{r=1}^{r=n}\left[-\lambda\,(n+1)\,x_r+(1+\lambda)\,s\right]\frac{du}{dx_r}=-\lambda\,(n+1)\,u+(1+\lambda)\,s.$$

7015. (By W. J. C. SHARP, M.A.)—Show that (1) if the two conics $S=0$ and $S'=0$ have double contact, $F\equiv\lambda S+\mu S'$, $\Phi\equiv\Lambda\Sigma+M\Sigma'$; if λ,μ have the same values as in (1), (2) the equations to the common tangents and common chord are

$$\lambda S-\mu S'=0 \text{ and } \left\{(\mu\Delta')^{\frac{1}{2}}S-(\lambda\Delta)^{\frac{1}{2}}S'\right\}^{\frac{1}{2}}=0;$$

(3) $\Lambda\Sigma-M\Sigma'=0$ represents the points of contact; (4) $(\Lambda)^{\frac{1}{2}}\Delta'\Sigma-(M)^{\frac{1}{2}}\Delta\Sigma'=0$ represents the intersection of the common tangents; also (5) furnish an answer to the query, in regard to Question 6204, on p. 75 of Vol. xxxiv. of the *Reprints*.

Solution by W. J. C. SHARP, M.A.

1, 2. If the conics be referred to their common self-conjugate triangle, so that $S\equiv ax^2+by^2+cz^2$ and $S'\equiv a'x^2+b'y^2+c'z^2$; then for double contact it is necessary and sufficient that two of the fractions $\frac{a'}{a}$, $\frac{b'}{b}$, $\frac{c'}{c}$ should be equal. Assume $\frac{a'}{a}=\frac{b'}{b}=r$, then the covariant

$$F\equiv aa'(bc'+b'c)\,x^2+bb'(ac'+a'c)\,y^2+cc'(a'b+ab')\,z^2$$
$$\equiv\frac{1}{r}\,a'b'c'\,(ax^2+by^2+cz^2)+r\,abc\,(a'x^2+b'y^2+c'z^2)$$
$$\equiv\lambda S+\mu S' \text{ if } \lambda=\frac{a'b'c'}{r},\ \mu=r\,abc,$$

therefore $\lambda S-\mu S'\equiv r\,abc'\,(ax^2+by^2+cz^2)-r\,abc\,(rax^2+rby^2+c\,z^2)$
$$\equiv r\,ab\,(c-rc)\,(ax^2+by^2),$$

and $(\mu\Delta')^{\frac{1}{2}}S-(\lambda\Delta)^{\frac{1}{2}}S' \equiv (r\Delta\Delta')^{\frac{1}{2}}S-\left(\frac{\Delta\Delta'}{r}\right)^{\frac{1}{2}}S'$

$$\equiv \left(\frac{\Delta\Delta'}{r}\right)^{\frac{1}{2}}(rS-S') = \left(\frac{\Delta\Delta'}{r}\right)^{\frac{1}{2}}(rc-c')z^2,$$

which prove the equations to the common tangents and chord.

3, 4. The other equation $\Phi \equiv \Delta\Sigma + M\Sigma'$ may be similarly deduced from the tangential equations to the conics, and this leads directly to the conditions given by Mr. Walker (Question 6204, *Reprint*, Vol. xxxiv., p. 75).

It is easy to show that the ratio $\lambda : \mu$ satisfies the equations

$$\Theta = \frac{\lambda}{\mu} + 2\left(\frac{\mu\Delta'}{\lambda\Delta}\right)^{\frac{1}{2}}, \quad \Theta' = \frac{\mu\Delta'}{\lambda\Delta} + 2\left(\frac{\lambda\Delta'}{\mu\Delta}\right)^{\frac{1}{2}}.$$

5. The query, propounded by Professor Wolstenholme in his solution to Question 6204, why cannot the conditions that two conics should have double contact be expressed in terms of their invariants, may be answered in the following manner, which shows that this is a consequence of the fact that it is not possible to express the conditions that a binary quartic should have two pairs of equal roots in terms of its invariants.

If $S=0$ and $S'=0$ be two conics having double contact, the quartic obtained by eliminating one of the coordinates will have two pairs of equal roots, and will differ from its Hessian by a multiplier only. But, since every covariant of a binary quartic can be expressed in terms of the quantic and its Hessian, every such covariant is in this case a multiple of the quartic. If now the coordinate already eliminated between S and S', be eliminated between $F=0$ (the covariant conic) and $S=0$, the eliminant will be a covariant of the binary quartic, and therefore a multiple of the eliminant of $S=0$ and $S'=0$, and therefore F must be of the form $hS+kS'$, the necessary and sufficient condition for double contact.

6957. (By Professor Wolstenholme, M.A.)—Prove that

$$\int_0^{\pi} \frac{\sin^2 nx}{\sin^2(1-n)x}\,dx = n\int_0^{\pi} \frac{\sin^2 x}{\sin^2(1-n)x}\,dx = \frac{n}{1-n}\int_0^{\pi} \frac{\sin(1+n)x}{\sin(1-n)x}\,dx \ldots(1),$$

$$\int_0^{\frac{\pi}{n}} \frac{\sin^2 nx}{\sin^2(1-n)x}\,dx = n\int_0^{\frac{\pi}{n}} \frac{\sin^2 x}{\sin^2(1-n)x}\,dx = \frac{n}{1-n}\int_0^{\frac{\pi}{n}} \frac{\sin(1+n)x}{\sin(1-n)x}\,dx \ldots(2),$$

if, in (1), $n>0<2$, and in (2) $n>\frac{1}{2}<\infty$.

Solution by J. Hammond, M.A.; Christine Ladd, B.A.; *and others.*

Only one of each set of these pretty equations is independent, the other being easily deducible from the relation $\sin^2\alpha-\sin^2\beta=\sin(\alpha+\beta)\sin(\alpha-\beta)$. Integrating by parts, we have

$$\int \sin^2\theta\,\mathrm{cosec}^2(1-n)x\,.\,dx = -\frac{\sin^2\theta}{1-n}\cot(1-n)x + \int\frac{\cot(1-n)x}{1-n}\sin 2\theta\,.\,d\theta,$$

and if in this, we take $\theta = x$, and the limits from 0 to π, so that

$$\int_0^{\pi} \frac{\sin^2 x}{\sin^2(1-n)x}\,dx = \frac{1}{1-n}\int_0^{\pi} \sin 2x \cot(1-n)x\,.\,dx\,;$$

then we obtain $\displaystyle\int_0^{\pi} \frac{\sin^2 x}{\sin^2(1-n)x}\,dx - \frac{1}{1-n}\int_0^{\pi} \frac{\sin(1+n)x}{\sin(1-n)x}$

$$= \frac{1}{1-n}\int_0^{\pi} \frac{\sin 2x \cos(1-n)x - \sin[2-(1-n)]x}{\sin(1-n)x}\,dx = \frac{1}{1-n}\int_0^{\pi} \cos 2x.dx = 0.$$

Again, in the same, take $\theta = nx$, and the limits from 0 to $\frac{\pi}{n}$, so that

$$\int_0^{\frac{\pi}{n}} \frac{\sin^2 nx}{\sin^2(1-n)x}\,dx = \frac{n}{1-n}\int_0^{\frac{\pi}{n}} \sin 2nx \cot(1-n)x\,.\,dx\,;$$

then we obtain $\displaystyle\int_0^{\frac{\pi}{n}} \frac{\sin^2 nx}{\sin^2(1-n)x}\,dx - \frac{n}{1-n}\int_0^{\frac{\pi}{n}} \frac{\sin(1+n)x}{\sin(1-n)x}\,dx$

$$= \frac{n}{1-n}\int_0^{\frac{\pi}{n}} \frac{\sin 2nx \cos(1-n)x - \sin(2n+1-n)x}{\sin(1-n)x}\,dx$$

$$= -\frac{n}{1-n}\int_0^{\frac{\pi}{n}} \cos 2nx\,.\,dx = 0 \quad\ldots\ldots\ldots\ldots(3).$$

As mentioned above, the other results of the question follow at once from (2) and (3); the limitation of the value of n is only introduced in order that the functions under the integral signs may remain finite for all values of x between the limits.

[If AP, BP be straight lines revolving about A, B, with angular velocities in the ratio $1 : n$, where n is a proper fraction, the equation of the locus of P is

$$r = a\frac{\sin n\theta}{\sin(1-n)\theta} \quad\text{or}\quad r = a\frac{\sin\frac{\theta}{n}}{\sin\left(\frac{\theta}{n}-\theta\right)},$$

according as A or B is taken for origin; and the area of the outer loop is

$$a^2\int_0^{\pi} \frac{\sin^2 n\theta}{\sin^2(1-n)\theta}\,d\theta \quad\text{or}\quad a^2\int_0^{n\pi} \frac{\sin^2\frac{\theta}{n}\,d\theta}{\sin^2\left(\frac{\theta}{n}-\theta\right)},$$

which latter $\displaystyle = na^2\int_0^{\pi} \frac{\sin^2\theta\,d\theta}{\sin^2(1-n)\theta}$ (writing $n\theta$ for θ).

So also the sum of the areas of the two loops is

$$a^2\int_0^{\frac{\pi}{n}} \frac{\sin^2 n\theta}{\sin^2(1-n)\theta}\,d\theta \quad\text{and}\quad a^2\int_0^{\pi} \frac{\sin^2\frac{\theta}{n}\,d\theta}{\sin^2\left(\frac{\theta}{n}-\theta\right)},$$

which latter $\quad = ra^2 \int_0^{\frac{\pi}{n}} \frac{\sin^2\theta\, d\theta}{\sin^2(1-n)\,\theta}.$

The true limits for n, in order that the subjects of integration may remain finite between the limits of integration, are $n > \frac{1}{2} < 2$.

In the same way, it may be proved that

$$\int_0^{\pi} \frac{\sin x \sin^3 nx}{\sin^6(1-n)\,x}\,dx = n \int_0^{\pi} \frac{\sin nx \sin^3 x}{\sin^3(1-n)\,x}\,dx$$

$$= \frac{n}{1-n} \int_0^{\pi} \frac{\sin x \sin nx \sin(1+n)\,x}{\sin^2(1-n)\,x}\,dx \quad \text{.........(4)},$$

$$\int_0^{\frac{\pi}{n}} \frac{\sin x \sin^3 nx}{\sin^3(1-n)\,x}\,dx = n \int_0^{\frac{\pi}{n}} \frac{\sin nx \sin^3 x}{\sin^3(1-n)\,x}\,dx$$

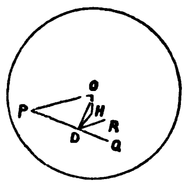
$$= \frac{n}{1-n} \int_0^{\frac{\pi}{n}} \frac{\sin x \sin nx \sin(1+n)\,x}{\sin^2(1-n)\,x}\,dx \quad \text{.........(5)},$$

if in (3) $n > 0 < 2$, and in (4) $n > \frac{1}{2}$ and $< \infty$.]

6852. (By Professor Cavallin, M.A.)—Within a sphere a point P is taken at random, and two other points Q, R are taken at random, Q in the sphere concentric with the given sphere and passing through P, and R in the sphere concentric with the given sphere and touching PQ; prove that, if a denotes the radius of the given sphere, p the perpendicular from its centre on the plane determined by the points P, Q, R, and n any positive number, the average value of p^n is

$$\frac{3.5.7.9}{(n+1)(n+3)(n+5)(n+7)(n+9)}\,a^n.$$

Solution by Professor Seitz, M.A.; G. Heppel, M.A.; *and others.*



Let OH be the perpendicular from O, the centre of the given sphere, on the plane PQR, and HD perpendicular to PQ. Then OD is perpendicular to PQ, and R is limited to the sphere whose radius is OD. Let $OP = x$, $PQ = y$, $DR = z$, $OH = p$, $\angle OPQ = \theta$, $\angle ODH = \phi$, and $\angle HDR = \psi$.

Then $OD = x\sin\theta$, $HD = x\sin\theta\cos\phi$, $p = x\sin\theta\sin\phi$; an element of the sphere at P is $4\pi x^2 dx$, at Q it is $2\pi y^2 \sin\theta\, d\theta\, dy$, and at R it is $z^2 \cos\psi\, d\psi\, dz$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and $\frac{1}{2}\pi$; of x, 0 and a; of y, 0 and $2x\cos\theta = y'$; of ψ, $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; and of z, 0 and $2x\sin\theta\cos\phi\cos\psi = z'$.

Hence, omitting constant factors common to numerator and denominator, we have, for the average value of p^n,

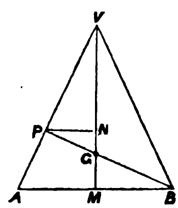
$$\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{a}\int_0^{y'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{z'} p^n \sin\theta\, d\theta\, d\phi\, x^2\, dx\, y^2\, dy \cos\psi\, d\psi\, z^2\, dz}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{a}\int_0^{y'}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{z'} \sin\theta\, d\theta\, d\phi\, x^2\, dx\, y^2\, dy \cos\psi\, d\psi\, z^2\, dz}$$

$$= \frac{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{a} \sin^{n+4}\theta \cos^3\theta\, d\theta \sin^n\phi \cos^3\phi\, x^{n+8}\, dx}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{a} \sin^4\theta \cos^3\theta\, d\theta \cos^3\phi\, d\phi\, x^8\, dx}$$

$$= \frac{\frac{a^{n+9}}{n+9}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)\left(\frac{1}{n+5}-\frac{1}{n+7}\right)}{\frac{a^9}{9}\left(1-\frac{1}{3}\right)\left(\frac{1}{5}-\frac{1}{7}\right)} = \frac{3.5.7.9}{(n+1)(n+3)(n+5)(n+7)(n+9)}a^n.$$

6814. (By Professor Hudson, M.A.) — Prove (1) the following construction for the centre of gravity of a right circular cone:—let AB be a diameter of the base, V the vertex, divide VA, VB in P, Q, in the ratio 3 : 2, where BP, AQ intersect in the centre of gravity; and show (2) that the same construction holds for a cone on an elliptic base.

Solution by T. Woodcock, B.A.; E. Rutter; *and others.*



Let PB meet the axis VM in G, and draw PN perpendicular to the axis; then NM = $\frac{2}{5}$ VM; but NG : GM = PN : BM = 3 : 5,

$\therefore$ GM : NM = 5 : 8, $\quad \therefore$ GM = $\frac{1}{4}$VM.

Therefore (1) G is the centre of gravity of a right circular cone; and, since the proof merely requires that M should be the middle point of each diameter AB of the base, it is (2) applicable also to a right cone on an elliptic base.

6425. (By W. J. C. Sharp, M.A.)—Prove that one and only one quadric surface can be drawn through a given twisted quintic curve.

Solution by the Proposer.

This is somewhat too generally stated; it should stand, "not more than one quadric can be drawn." This is a case of the general statement that "in general at most one n-ic surface can be drawn through a given twisted $\frac{1}{2}n(n+3)$-ic curve. For the curve cuts any arbitrary plane in $\frac{1}{2}n(n+3)$ points, which will in general determine one plane n-ic curve, and which must be the section of the n-ic surface if there be one. The exceptions are those noticed in Salmon's *Higher Curves*, and will occur when the curve is the partial intersection of two n-ic surfaces.

7073. (By Professor SYLVESTER, F.R.S.)—If from each point of inflexion tangents are drawn to a cubic curve, meeting it in 27 points, and from each of these tangents meeting it in 108 points, and again from each of these tangents meeting it in 432 points, and so on; prove that the $27 \cdot 4^{i-1}$ points so obtained by applying this process i times will be the complete system of points in which proper curves of the order 2^i can be drawn having $3 \cdot 2^i$ consecutive points in common with the cubic.

Solution by W. J. C. SHARP, M.A.; BELLE EASTON; *and others.*

If a, b, c, d, &c. be the $3 \cdot 2^i$ successive points in which a 2^i-ic curve meets the cubic, these form an entire intersection; and therefore (SALMON'S *Higher Plane Curves*, Art. 157) so do the $3 \cdot 2^{i-1}$ successive points a', b', &c., where ab, cd, &c. meet the cubic again, and these are the tangentials of the others. If this process be repeated i times, we arrive at three successive points forming an entire intersection, *i.e.*, at an inflexion.

7077. (By Professor MINCHIN, M.A.)—Prove that (1) the total energy of a rigid body is equal to the energy due to the velocity of translation of P + the energy due to the rotation round P, if P is any point on the quadric whose equation referred to the principal axes at G (the centre of mass) is $\lambda^2+\mu^2+\nu^2+a\lambda+b\mu+c\nu=0$, where (a, b, c) are the velocity components of G, and $\lambda \equiv \omega_2 z - \omega_3 y$ &c.; and (2) in uniplanar motion the locus of P is the circle described on GI as diameter (I ≡ instantaneous centre). [A property always enunciated as if belonging to G exclusively.]

Solution by Prof. TOWNSEND, F.R.S.; T. WOODCOCK, B.A.; *and others.*

That, for a rigid body in motion, the entire kinetic energy should be equal to the sum of its components due to the separate motions of translation and rotation at any point P of the mass, we must have, if uvw and pqr be the components of the two motions at the point, xyz the relative coordinates of the point with respect to the centre of inertia of the body, and $x+\xi$, $y+\eta$, $z+\zeta$ those of any element dm of its mass,

$$\Sigma\left\{\left[u+q(z+\zeta)-r(y+\eta)\right]^2+\&c.\right\}dm$$
$$=m\left[(u+qz-ry)^2+\&c.\right]+\Sigma\left[(q\zeta-r\eta)^2+\&c.\right]dm$$
$$=m\left[(u+qz-ry)^2+\&c.+(qz-ry)^2+\&c.\right];$$

therefore, at once, for every point xyz of the mass fulfilling the condition, we must have $$(u+qz-ry)(qz-ry)+\&c.=0,$$

and therefore, &c., as regards the first part of the question.

The equation $(qz-ry)^2+\&c.+u(qz-ry)+\&c.=0$, expressing that the square of the distance of xyz from the line $x:y:z=p:q:r$ varies as its distance from the plane $u(qz-ry)+\&c.=0$, shows consequently that the quadric in question is a circular cylinder passing through the line and touching the plane. Again, as the equation is also manifestly satisfied for every point xyz of the line

$$(u+qz-ry):(v+rx-pz):(w+py-qx)=p:q:r,$$

that is, of the axis of the instantaneous screw motion of the body, it shows consequently also that the cylinder passes through that line, and finally, as the plane of connection of the two lines

$x : y : z = p : q : r$ and $(u+qz-ry) : (v+rx-pz) : (w+pq-qx) = p : q : r$

is manifestly perpendicular to the plane

$$u(qz-ry)+v(rx-pz)+w(py-qx)=0,$$

it appears from it, lastly, that the plane of the two lines in question is a diametral plane of the cylinder, and that the lines themselves are consequently diametrically opposite generators of the surface. From these general results for motion in three dimensions, the second part of the question follows of course at once for the particular case referred to in its statement.

7079. (By the EDITOR.)—Prove that, if

$$ax+cy : by+dz = ay+cz : bz+dx = az+cx : bx+dy,$$

then (1) each ratio $= a+c : b+d$, and (2) $\Sigma x^2 = \Sigma(yz)$.

Solution by Prof. WOLSTENHOLME, M.A.; KATE GALE; *and others.*

If $$\frac{ax+cy}{by+dz}=\frac{ay+cz}{bz+dx}=\frac{az+cx}{bx+dy},$$

then each ratio $$=\frac{(ax+cy)+(ay+cz)+(az+cx)}{(by+dz)+(bz+dx)+(bx+dy)}=\frac{(a+c)(x+y+z)}{(b+d)(x+y+z)},$$

also $$=\frac{(ax+cy)+\omega(ay+cz)+\omega^2(az+cx)}{(by+dz)+\omega(bz+dx)+\omega^2(bx+dy)}=\frac{(a+c\omega^2)(x+\omega y+\omega^2 z)}{(b\omega^2+d\omega)(x+\omega y+\omega^2 z)},$$

if ω be one of the impossible cube roots of 1; and, similarly, each ratio

$$=\frac{(a+c\omega)(x+\omega^2 y+\omega z)}{(b\omega+d\omega^2)(x+\omega^2 y+\omega z)}.$$

Hence, either $x+y+z$, $x+\omega y+\omega^2 z$, or $x+\omega^2 y+\omega z=0$, or $x^3+y^3+z^3=3xyz$;

and each of the three ratios $=\dfrac{a+c}{b+d}$, or $\dfrac{a\omega+c}{b+d\omega^2}$, or $\dfrac{a\omega^2+c}{b+d\omega}$.

If all the symbols represent real quantities, we must have $x=y=z$, and each ratio $=\dfrac{a+c}{b+d}$; or $x+y+z=0$, and each ratio

$$=\frac{c}{b-d}=\frac{c-a}{b}=-\frac{a}{d}.$$

[Prof. WOLSTENHOLME remarks that, in the Mathematical Tripos Examination for 1857, Dr. CAMPION set the question to prove that $a^3+b^3+c^3=3abc$, if

$$\frac{a\sin^2\theta+b\sin^2\phi}{b\cos^2\theta+c\cos^2\phi}=\frac{b\sin^2\theta+c\sin^2\phi}{c\cos^2\theta+a\cos^2\phi}=\frac{c\sin^2\theta+a\sin^2\phi}{a\cos^2\theta+b\cos^2\phi}.$$

If we put each of the three ratios in Quest. 7079 $=\lambda$, we get at once

$$\begin{vmatrix} a, & c-\lambda b, & -\lambda d \\ -\lambda d, & a, & c-\lambda b \\ c-\lambda b, & -\lambda d, & a \end{vmatrix} = 0; \qquad \begin{vmatrix} x & y, & z \\ y, & z, & x \\ z, & x, & y \end{vmatrix} = 0,$$

or $a^3+(c-\lambda b)^3-\lambda^3 d^3+3\lambda ad\,(c-\lambda b) = 0$, which gives the three factors

$$a+c-\lambda b-\lambda d, \quad a+\omega\,(c-\lambda b)-\lambda\omega^2 d, \quad a+\omega^2\,(c-\lambda b)-\lambda\omega d,$$

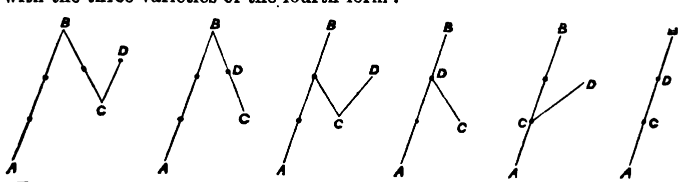
or gives for the three equal ratios the three values given above; the other determinant giving $x^3+y^3+z^3 = 3xyz$.]

6709. (By Dr. MACFARLANE, D.Sc., F.R.S.E.)—A is a great-grandchild of B, B is a grandparent of C, and C is a child of D. What is the relationship of A to D, expressed independently of B and C?

Solution by the PROPOSER.

The data are $A = c^3B$, $B = c^{-2}C$, $C = cD$. Hence $A = c^{3-2+1}D$; that is, A is a great-grandchild of a grandparent of a child of D. This is the general expression for the resulting relationship, and it can assume one or other of four irreducible forms. In the first place, the index $3-2$ may not cancel at all, or it may cancel once, giving $2-1$, or it may cancel twice, giving 1. Next, add on the $+1$ to each of these. In the first case, we have $3]-2+1$, which may not cancel, or it may cancel once; in the second case, we have $2]-1+1$, which may not cancel, or it may cancel once. In the third case, no cancelling is possible. Thus we obtain four, and only four, irreducible forms. Hence $A = c^{3-2+1}D$; or $A = c^{3-1}D$; or $A = c^{2-1+1}D$; or $A = c^2D$, where each relationship expression must be interpreted in its irreducible meaning. These equations mean that A is a grandchild of a brother-in-law or sister-in-law of D; or A is a grand-nephew or grandniece of D; or A is a grandchild of a consort of D; or A is a grandchild of D.

These forms are represented graphically in the accompanying figure, with the three varieties of the fourth form :—



Form 1. Form 2. Form 3. Form 4.

A stroke of unit length in an upward direction represents c, and a similar stroke in a downward direction represents c^{-1}. The first form represents the general expression for the relationship, provided it be understood that two or more of the lines may coincide. In the case of a figure representing an irreducible relationship no pair of lines can coincide except the two lines be drawn as coincident.

7032. (By Professor Johnson, M.A.)—If three triangles have, when taken in pairs, a common axis of homology, prove that (1) the three centres of homology are in a straight line; (2) reciprocally, if three triangles have, when taken in pairs, a common centre of homology, the three axes of homology pass through a common point.

Solution by Prof. Genese, M.A.; Prof. Matz, M.A.; *and others.*

Project the axis of homology to infinity. The three triangles will now have their sides parallel. Let A_1, A_2, A_3 be three corresponding vertices; $m_1 : m_2 : m_3$ the ratios of the linear dimensions of the triangles. Then C_1, the centre of homology of second and third triangles, lies on A_2A_3, so that $A_2C_1 : A_3C_1 = m_2 : m_3$, and similarly for C_2, C_3. Hence, by the theory of transversals, C_1, C_2, C_3 are in a straight line, &c.

6869. (By A. McIntosh, M.A.)—Prove that (1) only two real tangents can be drawn from any point on the curve itself to a lemniscate; (2) the same line passes through the two real and the two imaginary points of contact of the four tangents that are analytically possible; (3) the following construction gives this line,—draw a radius-vector from the real node to the point from which the tangents are required to be drawn, produce it backwards one-half of its length, and through this extremity draw a line perpendicular to it, and this last will be the line sought; (4) hence a chord joining the points of contact of tangents drawn from any point on the curve touches an equilateral hyperbola, which has the same axis and the same foci as the lemniscate, which passes through the points of contact of the double tangents (which, of course, touches all the tangents that can be drawn from the three nodes to the curve); (5) to a lemniscate is drawn a tangent, which meets the curve again in two points, if a circle be drawn through the node and these two points, its centre lies on the lemniscate and its radius is equal to the radius vector from the node to the point of contact of the tangent; (6) if four tangents be drawn from any point (x', y', z') on the curve $ax^{-2}+by^{-2}+cz^{-2}=0$ to touch the curve elsewhere, the four points of contact lie on the line $xx'^{-1}+yy'^{-1}+zz'^{-1}=0$, which touches the conic $a^{-1}x^2+b^{-1}y^2+c^{-1}z^2=0$, which passes through the eight points of contact of bitangents to the curve; (7) the above trinodal quartic includes the case of the lemniscate.

Solution by the Proposer.

The equation of the lemniscate being $(x^2+y^2)^2 = a^2(x^2-y^2)$, the equation of a tangent at any point (x', y') is

$$(x-x')\left[4x'(x'^2+y'^2)-2a^2x'\right] + (y-y')\left[4y'(x'^2+y'^2)+2a^2y'\right] = 0.$$

This may be reduced by the equation of the curve to

$$x\left[2x'(x'^2+y'^2)-a^2x'\right]+y\left[2y'(x'^2+y'^2)+a^2y'\right] = a^2(x'^2-y'^2).$$

Hence the coordinates of the point of contact of a tangent from the point (x', y') on the curve to the curve are given by the equation

$$x'\left[2x(x^2+y^2)-a^2x\right]+y'\left[2y(x^2+y^2)+a^2y\right] = a^2(x^2-y^2).$$

The coordinates of any point on the lemniscate may be represented by $a\dfrac{z+z^3}{1+z^4}$ and $a\dfrac{z-z^3}{1+z^4}$ where z is variable. Substitute, then, in the above equation for x and y, $a\dfrac{z+z^3}{1+z^4}$ and $a\dfrac{z-z^3}{1+z^4}$, and for x' and y', $a\dfrac{z'+z'^3}{1+z'^4}$ and $a\dfrac{z'-z'^3}{1+z'^4}$ respectively, and after reducing the resulting equation and dividing by $(z-z')^2$, it will be found to be

$$z^4z'+2z^3z'^2+2z+z'=0 \text{ or } x\,(z'^{-1}+z')+y\,(z'^{-1}-z')+a=0,$$

x being $a\dfrac{z+z^3}{1+z^4}$, and y, $a\dfrac{z-z^3}{1+z^4}$. This equation evidently denotes a line passing through the four points of contact of tangents from the points z'. It is perpendicular to the line $\dfrac{x}{z'+z'^3}=\dfrac{y}{z'-z'^3}$, which passes through the origin and the point z'; also the distance of it from the origin is

$$\frac{a}{\left\{2\left(\frac{1}{z'^2}+z'^2\right)\right\}^{\frac{1}{2}}}=\frac{z'a\,2^{\frac{1}{2}}}{2\,(1+z'^4)^{\frac{1}{2}}}.$$

But the distance of the point z' from the origin is

$$a\left\{\left(\frac{z'+z'^3}{1+z'^4}\right)^2+\left(\frac{z'-z'^3}{1+z'^4}\right)^2\right\}^{\frac{1}{2}}=\frac{az'\,2^{\frac{1}{2}}}{(1+z'^4)^{\frac{1}{2}}};$$

therefore the line joining the origin to the point from which the tangents are drawn is perpendicular to the line joining the points of contact, and if produced back to meet the latter line, the origin or node is a point of trisection. It is not difficult to see, now, that the point of intersection of these two lines lies on the lemniscate $(x^2+y^2)^2=\dfrac{a^2}{4}(x^2-y^2)$, and the envelope of the line joining the points of contact, being the negative pedal of this lemniscate, is the hyperbola $x^2-y^2=\dfrac{a^2}{4}$.

This equilateral hyperbola has the nodal or inflexional tangents for asymptotes, and its foci are at the distances $\dfrac{a}{\sqrt{2}}$ and $-\dfrac{a}{\sqrt{2}}$ from the origin, and therefore coincide with the foci of the lemniscate. The points in which it cuts the lemniscate are given by $r^2=a^2\cos 2\theta$ and $r^2\cos 2\theta=\dfrac{a^2}{4}$, therefore $\dfrac{1}{\cos 2\theta}=4\cos 2\theta$, *i.e.*, $\cos^2 2\theta=\frac{1}{4}$.
Hence $\cos 2\theta=\pm\frac{1}{2}$, therefore $2\theta=60^\circ$ and $\theta=30^\circ$, but this is the point of contact of a bitangent.

Consider the point of contact T of any tangent and the two points R and S in which it cuts the curve. Let N be the node. By what has just been proved, the line bisecting NR perpendicularly is parallel and similarly situated with respect to one of the loops to the line joining the points of contact of tangents from R to the curve. Hence it cuts the curve in points equally distant from the node, and consequently cuts it in one point the distance of which from the node is equal to NT. The same thing may be said of the line bisecting NS perpendicularly and therefore this line meets the curve where the line bisecting NR perpendicularly meets it. The

distance of this point, which is the centre of the circle circumscribing the triangle NRS, from the node is equal to NT.

To prove that only two real tangents can be drawn from a point on the curve, it is only necessary to remark that the invariant I of the biquadratic $z^4z' + 2z^3z' + 2z + z' = 0$ vanishes. The reducing cubic of this equation, by Euler's method, may then be brought to the form $\lambda^3 + 2J = 0$. This has only one real root, therefore two roots of the biquadratic are real and two are imaginary. This is a point sufficiently evident from the figure of the curve, but it is interesting to see how it comes out analytically.

The equation of a tangent at the point (x', y', z') to the curve $\dfrac{a}{x^2} + \dfrac{b}{y^2} + \dfrac{c}{z^2} = 0$ may be put in the form $\dfrac{ax}{x'^3} + \dfrac{by}{y'^3} + \dfrac{cz}{z'^3} = 0$. Hence, if this line touches the curve at (x, y, z) and intersects it at (x', y', z'), the equation connecting the points will be $\dfrac{ax'}{x^3} + \dfrac{by'}{y^3} + \dfrac{cz'}{z^3} = 0$.

Any point on the curve will be given (i being $= \sqrt{-1}$) by

$$\frac{a^{\frac{1}{2}}z}{c^{\frac{1}{2}}x} = i\cos\theta \text{ and } \frac{b^{\frac{1}{2}}z}{c^{\frac{1}{2}}y} = i\sin\theta.$$

Substituting then these values in the foregoing equation, together with $\dfrac{a^{\frac{1}{2}}z'}{c^{\frac{1}{2}}x'} = i\cos\theta'$ and $\dfrac{a^{\frac{1}{2}}z'}{c^{\frac{1}{2}}y'} = i\sin\theta'$, we easily get $\dfrac{\cos^3\theta}{\cos\theta'} + \dfrac{\sin^3\theta}{\sin\theta'} = 1$;

$\therefore\ \cos^3\theta\sin\theta' + \sin^3\theta\cos\theta' = \cos\theta'\sin\theta' = \cos\theta'\sin\theta'(\cos^2\theta + \sin^2\theta)$.

Hence $\sin\theta'\cos^2\theta(\cos\theta - \cos\theta') + \cos\theta'\sin^2\theta(\sin\theta - \sin\theta') = 0$.

Hence
$$\tan\theta'\cot^2\theta\tan\tfrac{1}{2}(\theta + \theta') = 1,$$

therefore
$$\frac{\tan\frac{1}{2}(\theta + \theta')}{\tan\theta'} = \frac{\tan^2\theta}{\tan^2\theta'},$$

$$\frac{\sin\frac{1}{2}(\theta - \theta')}{\cos\frac{1}{2}(\theta + \theta')\cos\theta'\tan\theta'} = \frac{\sin(\theta - \theta')\sin(\theta + \theta')}{\cos^2\theta\cos^2\theta'\tan^2\theta'},$$

$$1 = \frac{2\cos\frac{1}{2}(\theta - \theta')\cos\frac{1}{2}(\theta + \theta')\sin(\theta + \theta')}{\cos^2\theta\sin\theta'},$$

$$\cos^2\theta\sin\theta' = \cos\theta\sin\theta\cos\theta' + \sin\theta'\cos^2\theta + \sin\theta\cos^2\theta' + \sin\theta'\cos\theta'\cos\theta$$

or
$$\cos\theta\sin\theta + \sin\theta\cos\theta' + \sin\theta'\cos\theta = 0.$$

When this is transformed, by resubstituting for $\cos\theta$, $\sin\theta$, &c., in terms of x, y, z, &c., it becomes $\dfrac{x}{x'} + \dfrac{y}{y'} + \dfrac{z}{z'} = 0$.

This is the equation of a line which clearly passes through the four points of contact of four tangents from (x', y', z') to the curve.

Since also $\dfrac{a}{x'^2} + \dfrac{b}{y'^2} + \dfrac{c}{z'^2} = 0$, it touches the conic $\dfrac{x^2}{a} + \dfrac{y^2}{b} + \dfrac{z^2}{c} = 0$, which passes through the eight points of contact of bitangents to the curve

$$\frac{a}{x^2} + \frac{b}{y^2} + \frac{c}{z^2} = 0.$$

This may also be proved without the use of trigonometrical functions. Let $\dfrac{z}{x} = \lambda$ and $\dfrac{z}{y} = \mu$, with similar substitutions for $\dfrac{z'}{x'}$ and $\dfrac{z'}{y'}$.

Then we have for the tangent

$$\frac{a\lambda^3}{\lambda'}+\frac{b\mu^3}{\mu'}+c=0 \quad\text{.................................(1).}$$

Also $\qquad a\lambda^2+b\mu^2+c=0 \quad\text{and}\quad a\lambda'^2+b\mu'^2+c=0$.............(2, 3).

From (1), (2), and (2), (3),

$$a\lambda^2\left(\frac{\lambda}{\lambda'}-1\right)=-b\mu^2\left(\frac{\mu}{\mu'}-1\right),\quad a\,(\lambda^2-\lambda'^2)=-b\,(\mu^2-\mu'^2).$$

Dividing, we have $\qquad \dfrac{(\lambda+\lambda')\,\lambda'}{\lambda^2}=\dfrac{(\mu+\mu')\,\mu'}{\mu^2}.$

When this equation is cleared from fractions, it will be seen to be divisible by $\lambda'\mu-\lambda\mu'$, and the result will be $\lambda\mu+\lambda'\mu+\lambda\mu'=0$, which comes to the same thing as before.

5149. (By S. Tebay, B.A.)—A solid, having a plane base, rests on a smooth horizontal plane, and a heavy particle descends in a vertical plane passing through the centre of gravity in a curvilinear groove between two given points in the solid; find the brachystochrone.

Solution by the Proposer.

Let the higher point be the origin, m the mass of the particle, m' the mass of the solid, xy the coordinates of m at time t, x' the abscissa of the centre of gravity, and h the initial value of x' Then, by the principle of *vis viva*,
$$m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}+m'\left(\frac{dx'}{dt}\right)^2=2gmy.$$

But $\quad mx+m'x'=m'h$; therefore, putting $1+\dfrac{m'}{m}=n^2$, we have

$$n^2\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=2gy;\quad\text{therefore}\quad t=\int\left\{\frac{p^2+n^2}{2gy}\right\}^{\frac12}dx,$$

and
$$V=\left\{\frac{p^2+n^2}{y}\right\}^{\frac12}=Pp+\frac{1}{c}.$$

But $\quad P=\dfrac{p}{[y\,(p^2+n^2)]^{\frac12}};\qquad$ therefore $\quad p^2=\left(\dfrac{dy}{dx}\right)^2=n^2\left(\dfrac{c^2n^2}{y}-1\right),$

and
$$nx=\tfrac12 c^2n^2\operatorname{vers}^{-1}\frac{2y}{c^2n^2}-(c^2n^2y-y^2)^{\frac12}.$$

Since the solid has slipped back through the space $h-x'=\dfrac{m}{m'}x$, if we write $\dfrac{m}{m'}x+x=n^2\dfrac{m}{m'}x$ for x, we get the equation to the groove, namely,

$$n^3\frac{m}{m'}x=\tfrac12 c^2n^2\operatorname{vers}^{-1}\frac{2y}{c^2n}-(c^2n^2y-y^2)^{\frac12}.$$

If a, b be the coordinates of the second point, we have

$$n^3\frac{m}{m'}a=\tfrac12 c^2n^2\operatorname{vers}^{-1}\frac{2b}{c^2n^2}-(c^2n^2b-b^2)^{\frac12},$$

which determines c.

7012. (By J. J. WALKER, M.A.)—Show that (1) the squares of tangents drawn from an external point (x', y') to an ellipse are, if $s' = b^2x'^2 + a^2y'^2 - a^2b^2$, given by the equation

$$(b^2x'^2 + a^2y'^2)^2 t^4 - 2\left[s'(x'^2+y'^2) + b^4x'^2 + a^4y'^2\right] s't^2 + \left[c^4 - 2c^2(x'^2-y'^2) + (x'^2+y'^2)\right]^2 s'^2 = 0;$$

(2) the lengths t, t' of the tangents are given linearly by

$$(b^2x'^2 + a^2y'^2)\, t^2 - \frac{2a'b'}{k} s't + (a'^2 + b'^2 - x'^2 - y'^2)\, s' = 0;$$

viz., $\quad + t' = \dfrac{2a'b}{k} \sin^2 \tfrac{1}{2}(\theta-\theta'), \qquad t - t' = \dfrac{c^2k}{a'b'} \sin(\theta+\theta') \tan\tfrac{1}{2}(\theta-\theta'),$

where a', b' are the semi-axes of the confocal ellipse through (x', y'); $k^2 = a'^2 - a^2 = b'^2 - b^2$; and θ, θ' are the excentric angles of the points of contact of the tangents t, t' respectively.

Solution by Rev. T. R. TERRY, M.A.; Prof. WOLSTENHOLME; *and others.*

1. From the usual quadratic for the length of the intercept

$$\rho^2(a^2\sin^2\phi + b^2\cos^2\phi) + 2\rho(a^2y'\sin\phi + b^2x'\cos\phi) + s' = 0,$$

we see that for a tangent we must have

$$s'(a^2\sin^2\phi + b^2\cos^2\phi) = (a^2y'\sin\phi + b^2x'\cos\phi)^2 \quad \text{............(1)},$$

$$a^2\sin^2\phi + b^2\cos^2\phi = \frac{s'}{t^2} \quad \text{..............................(2)};$$

$$\therefore \quad a^2b^2t^2(a^2y'^2 - b^2x'^2) + c^2s'^2 - a^4y'^2s' + b^4x'^2s' = 2a^2b^2x'y'c^2t^2\sin\phi\cos\phi,$$

whence squaring we get the first result.

2. If we denote this and the second result by

$$A^2t^4 - 2Bt^2 + C = 0, \quad \text{and} \quad At^2 - 2B't + C' = 0,$$

the conditions that the roots of the second equation should be the two positive roots of the first equation are $B = 2B'^2 - AC'$ and $C = C'^2$, which are readily proved to be true by means of the equation

$$k^4 + k^2(a^2 + b^2 - x'^2 - y'^2) - s' = 0.$$

Again, since $s' = (b^2x'^2 + a^2y'^2)\sin^2\frac{1}{2}(\theta-\theta')$, it follows from the second equation that $t + t' = \dfrac{2a'b'}{k}\sin^2\frac{1}{2}(\theta-\theta')$; but directly from a figure we see that $t^2 - t'^2 = 2c^2\sin^2\frac{1}{2}(\theta-\theta')\sin(\theta+\theta')\tan\frac{1}{2}(\theta-\theta')$, whence the last result.

7054. (By Professor MALET, M.A.)—If from any point on the outer of two similar and coaxal ellipsoids a tangent cone be drawn to the inner, prove that the volume contained between the surfaces of the cone and either of the ellipsoids is constant.

Solution by T. WOODCOCK, B.A.; J. O'REGAN; *and others.*

The theorem is evidently true for two concentric spheres, and it is of the "projective" kind. Now two concentric spheres project orthogonally into two similar coaxal ellipsoids. Therefore, &c.

7068. (By W. J. C. SHARP, M.A.)—Show that, if n be any integer, n^5-n is divisible by 30, and by 240 when n is odd.

Solution by EMMA ESSENNELL; J. S. JENKINS; *and others.*

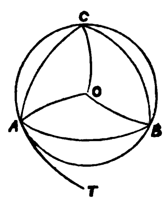
Since $n^5-n = n(n-1)(n+1)(n^2+1)$, and n must be of the form $5m$, $5m\pm1$, $5m\pm2$; if we put either of these values for n in the given expression, either of the factors becomes a multiple of 5; next $(n-1)\,n\,(n+1)$ three consecutive integers must be divisible by 6; thus n^5-n is divisible by $6\times5 = 30$.

Next, it will be sufficient to show that, if n is odd, n^5-n is divisible by 16, having already shown that in every case it is divisible by 15.

Put $n=2m+1$; we obtain $8\,(2m+1)\,m\,(m+1)\,(2m^2+2m+1)$; $m\,(m+1)$ is evidently divisible by 2, therefore n^5-n is divisible by 16 when n is odd, and thus by $16\times15 = 240$.

4784. (By H. HART.)—If ABC be a spherical triangle, and AT the tangent arc to the circumscribing small circle, prove that the angle BAT is equal to the angle C in the alternate segment minus half the spherical excess of the triangle.

Solution by J. HAMMOND, M.A.; BELLE EASTON; *and others.*



Let E_1, E_2, E_3 denote respectively the spherical excess of the triangles BOC, COA, AOB; then

$$E_1+E_2+E_3 = E,$$

the spherical excess of ABC; $\text{BAT}+\text{OAB} = \frac{1}{2}\pi$;

$$2\text{OAB}+\text{AOB} = \pi+E_3,\quad 2\text{OCA}+\text{AOC} = \pi+E_2,$$

$$2\text{OCB}+\text{BOC} = \pi+E_1.$$

Whence $\qquad 2\text{OAB}+2\text{C}+2\pi = 3\pi+E;$

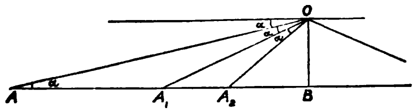
therefore $\qquad \text{OAB} = \frac{1}{2}\pi-\text{C}+\frac{1}{2}E,$

and $\qquad \text{BAT} = \text{C}-\frac{1}{2}E.$

6841. (By Rev. A. J. C. ALLEN, M.A.)—From a point O a straight line OA is drawn, making an angle $\alpha \left(<\dfrac{\varpi}{n+1}\right)$ with a fixed straight line AB, and n other straight lines $OA_1, OA_2 \ldots OA_n$ are drawn to it, making the angles $AOA_1, A_1OA_2\ldots$ all equal and each $=\alpha$; if $R_1, R_2 \ldots R_n$ be the radii of the circles circumscribing the triangle OAA_1, OA_1A_2; prove that, if a be the perpendicular from O on AA_n, then

$$R_1+R_2+\ldots+R_n = -\frac{a\sin n\alpha}{\sin 2\alpha\cos(n+1)\alpha}.$$

Solution by T. WOODCOCK, B.A.; G. M. REEVES, M.A.; *and others.*



We have

$R_1 = \frac{1}{2} AA_1 \operatorname{cosec} \alpha,$

$R_2 = \frac{1}{2} A_1A_2 \operatorname{cosec} \alpha,$

and so on,

$$\therefore R_1 + R_2 + \ldots + R_n = \tfrac{1}{2} AA_n \operatorname{cosec} \alpha$$

$$= \tfrac{1}{2} a \left\{ \cot \alpha - \cot (n+1) \alpha \right\} \operatorname{cosec} \alpha = \frac{-a \sin n\alpha}{\sin 2\alpha \cos (n+1) \alpha}.$$

7063. (By R. F. SCOTT, M.A.)—A and B are partners in a business in which their interests are in the ratio of a to b. They admit C to the partnership (without altering the capital) in such a way that the interests of the three partners in the business are then equal. C pays £c for the privilege; how is this to be divided between A and B?

Solution by EMMA ESSENNELL; E. RUTTER; *and others.*

The capital is evidently £$3c$, and therefore the values of A's and B's shares at first are £$\dfrac{3ac}{a+b}$ and £$\dfrac{3bc}{a+b}$ respectively, and the sums they should receive £$\left(\dfrac{3ac}{a+b} - c\right)$ and £$\left(\dfrac{3bc}{a+b} - c\right)$, which are as $2a - b : 2b - a$. If, therefore, a exceed $2b$, B will have to pay a sum of £$\dfrac{a-2b}{a+b} c$ instead of receiving anything.

6683. (By G. F. WALKER, M.A.)—If n particles of masses $m_1, m_2 \ldots m_n$ respectively, are rigidly connected together, and are capable of motion one on each of a series of smooth concentric circles in a vertical plane of radii $a_1, a_2 \ldots a_n$ respectively; show that the time of a small oscillation about the position of stable equilibrium under the action of gravity is

$$2\pi \left(\frac{\Sigma m_r a_r^2}{g \left(\Sigma m_r^2 a_r^2 + 2 S m_r m_s a_r a_s \cos \theta_{rs}\right)^{\frac{1}{2}}} \right)^{\frac{1}{2}},$$

where Σ denotes summation from $r = 1$ to $r = n$, and S denotes summation for all different pairs of values of r and s, and θ_{rs} is the angle subtended at the common centre by the line joining m_r and m_s.

Solution by CHRISTINE LADD, B.A.; *the* PROPOSER; *and others.*

Let θ be the angle the line joining m_1 to the centre makes with the vertical, and let α_r be the angle subtended at the centre by the line join-

ing m_r to m_{r+1}, and let s_r denote $a_1+a_2+\dots a_{r-1}$. Equation of motion is

$$\Sigma m\left(a\frac{d\theta}{dt}\right)^2 = C+2g\Sigma ma\cos(\theta+s),\ \therefore\ \Sigma ma^2\frac{d^2\theta}{dt^2} = -g\Sigma ma\sin(\theta+s),$$

Position of equation is given by $\Sigma ma\sin(\theta_0+s)=0$, and consequently

$$\Sigma ma^2\frac{d^2\phi}{dt^2} = -g\Sigma ma\cos(\theta_0+s)\,\phi = -g\left\{[\Sigma ma\cos s]^2+[\Sigma ma\sin s]^2\right\}^{\frac{1}{2}}\phi,$$

and required time $= 2\pi\left(\dfrac{\Sigma ma^2}{g\left\{[\Sigma ma\cos s]^2+[\Sigma ma\sin s]^2\right\}^{\frac{1}{2}}}\right)^{\frac{1}{2}}$ = &c., as given.

6786. (By Asutosh Mukhopâdhyây.)—Prove (1) geometrically that the rectangle contained by the perimeters of an acute-angled triangle and its pedal triangle, is equal to twice the sum of the rectangles contained by the perpendiculars, two and two; and (2) investigate whether the proposition holds for obtuse-angled triangles.

Solution by D. Edwardes, B.A.; Belle Easton; *and others.*

The lines joining the vertices of a triangle to the centre of circumscribed circle are perpendicular to the sides of the pedal; therefore $\Delta=\frac{1}{2}R$ (perimeter of pedal triangle). Again, $\Sigma p_1p_2=\dfrac{\Delta^2}{abc}$ (perimeter of original triangle), and $R=abc\div 4\Delta$, therefore &c.

6741. (By Professor Hoover.)—Show that the average area of circles whose diameters are the focal chords of an ellipse of semi-axes a, b, is $\dfrac{b\pi}{2a}(a^2+b^2)$.

Solution by Kate Gale; D. Edwardes; *and others.*

Required average $=\displaystyle\int_0^{\frac{1}{2}\pi}\pi\rho^2 d\theta \div \int_0^{\frac{1}{2}\pi} d\theta$ and $\rho=\dfrac{l}{1-e^2\cos^2\theta}$,

therefore $\quad\Delta = 2l^2\displaystyle\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-e^2\cos^2\theta)^2} = 2l^2\int_0^{\infty}\frac{(1+z^2)\,dz}{(1-e^2+z^2)^2}$

$$= 2l^2\int_0^\infty\frac{dz}{1-e^2+z^2}+2e^2l^2\int_0^\infty\frac{dz}{(1-e^2+z^2)^2} = \frac{2l^2}{(1-e^2)^{\frac{1}{2}}}\left[\tan^{-1}\frac{z}{(1-e^2)^{\frac{1}{2}}}\right]_0^\infty$$

$$+\frac{2e^2l^2}{2(1-e^2)^{\frac{1}{2}}}\left[\frac{1}{1-e^2}\times\tan^{-1}\frac{z}{(1-e^2)^{\frac{1}{2}}}+\frac{z}{(1-e^2)^{\frac{1}{2}}(1-e^2+z^2)}\right]_0^\infty$$

$$=\frac{b^3\pi}{a}+\frac{e^2ab\pi}{2}=\frac{b\pi}{2a}(a^2+b^2).$$

7003. (By Prof. NASH, M.A.)—The reciprocal polar of a given central conic with respect to a circle is similar to the given conic; prove that (1) the locus of the centre of the circle is a bicircular quartic having a node at the centre, and foci coinciding with the foci of the conic; (2) if the given conic is an ellipse, the nodal tangents are the equi-conjugate diameters; (3) if the conic is an hyperbola, the node is a conjugate point; (4) in the latter case the eccentricity of the hyperbola must be $< 2^{\frac{1}{2}}$ for the locus to be a real curve.

Solution by Prof. TOWNSEND, F.R.S.; Prof. MATZ, M.A.; *and others.*

The asymptotic angle of the reciprocal polar of a conic with respect to a circle being equal to that determined by the two tangents to the conic from the centre of the circle, the required locus, for a given central conic of semi-axes a and b, is consequently that of the intersections of its tangents at the angle of its asymptotes; the equation of which is found immediately as follows:—

Since, for tangents intersecting at any constant angle ϕ, the equation of the locus of their intersections (see SALMON's *Conic Sections*, 5th edit., art. 169) is $[(x^2+y^2)-(a^2+b^2)]^2\tan^2\phi = 4[b^2x^2+a^2y^2-a^2b^2]$, which represents in all cases a bicircular quartic having quadruple contact with the conic at the four imaginary points of its intersection with its orthocycle, the four tangents at which to both curves intersect in pairs at the two circular points at infinity and at the four foci of the conic, which latter in consequence are also foci of the quartic; therefore, for tangents intersecting at the asymptotic angle of the conic, for which

$$(a^2+b^2)^2\tan^2\phi + 4a^2b^2 = 0,$$

the equation is

$$a^2b^2[(x^2+y^2)-(a^2+b^2)]^2 + (a^2+b^2)^2[b^2x^2+a^2y^2-a^2b^2] = 0,$$

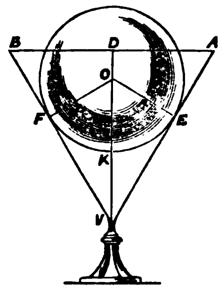
or its obvious equivalent on reduction

$$a^2b^2(x^2+y^2)^2 + (a^4-b^4)(a^2y^2-b^2x^2) = 0,$$

which shows that in that case the locus has, in addition, a double point at the origin, the tangents at which are given by the equation $a^2y^2-b^2x^2=0$, and are consequently real for the ellipse and coincide with the equiconjugate diameters of the curve, and imaginary for the hyperbola; and is itself real or imaginary in the latter case, according as b^2, which is then negative, is less or greater in absolute magnitude than a^2, that is, according as (a^4-b^4)—see last equation—is positive or negative. And therefore, &c., as regards the several parts of the equation.

7108. (By the EDITOR.)—Into a full conical wine-glass whose depth is a and generating angle α, there is dropped a spherical ball that causes the greatest overflow; show that (1) the radius of the ball is $\dfrac{a\sin\alpha}{\sin\alpha+\cos 2\alpha}$, and (2) when $\alpha = 30°$, the radius of the ball is $\frac{1}{2}a$, and its centre is at the middle of the top of the wine-glass.

Solution by J. W. SHARPE, M.A.; SARAH MARKS; *and others.*



1. Let $OE = OF = OK =$ radius of the ball $= x$; $OD = y$; $DK = x+y =$ length of the vertical diameter below the mouth of the cone; $V =$ volume of the ball below this level; then we have
$V = \frac{2}{3}\pi x^3 + \pi y (x^2 - \frac{1}{3}y^2)$ and $x \operatorname{cosec} \alpha + y = a$,
If V be a maximum or a minimum, we have, by differentiation,

$$6x(x+y)\,dx + 3(x^2-y^2)\,dy = 0$$

and $dx + \sin\alpha\,dy = 0$; hence for one condition $x + y = 0$, which corresponds to the case of the ball just touching the surface AB of the fluid,—a case excluded by the conditions of the question. Rejecting this factor, we get $2x\,dx + (x-y)\,dy = 0$, and $dx + \sin\alpha\,dy = 0$. Eliminating the ratio $dy : dx$ we obtain

$$2x \sin\alpha - x + y = 0, \text{ where } y = a - x \operatorname{cosec}\alpha; \text{ hence } x = \frac{a \sin\alpha}{\sin\alpha + \cos 2\alpha}.$$

2. When $\alpha = 30°$, we have $x = \frac{1}{2}a$, and $y = 0$.

6195. (By Professor COCHEZ.)—Démontrer que le lieu des foyers des hyperboles ayant un sommet donnée et une asymptote également donnée est une Strophoïde droite.

Solution by Prof. SCOTT, M.A.; W. J. C. SHARP, M.A.; *and others.*

If the given asymptote be taken as axis of x, the perpendicular from the given vertex upon it as axis of y, so that the vertex is $(0, k)$; and if $y = \tan 2\alpha\,(x+c)$ be the other asymptote, so that $(-c, 0)$ is the centre, then $y = \tan\alpha\,(x+c)$ is the axis, and $k = c\tan\alpha$ and at the focus $x + c = c\sec\alpha$ and $y = \tan\alpha\,(x+c)$; therefore $\cos\alpha = \dfrac{k}{y}$ and $x\sin\alpha = k(1-\cos\alpha)$; therefore $x^2(y+k) = k^2(y-k)$ or $(x^2-k^2)(y+k) + 2k^3 = 0$, the locus required, a cubic with three inflexional asymptotes, and no node.

7018. (By W. R. WESTROPP ROBERTS, M.A.)—Determine the locus of the point of contact of a geodesic tangent on an ellipsoid, touching a fixed line of curvature, with a series of sphero-conics on the same surface.

Solution by Professor TOWNSEND, F.R.S.

Denoting by O the centre of the ellipsoid, by abc its axes of figure, by δ

the parameter of the fixed line of curvature, by xyz the coordinates of the point of contact R with the variable sphero-conic of any of the four geodesics touching it and the line, by r the radius OR, by p the perpendicular OP from O on the tangent plane at R, by q the distance PR, by s the radius OS perpendicular to the plane POR and therefore parallel to the sphero-conic tangent at R, by $\alpha\beta\gamma$ the direction angles of OP, and by $\lambda\mu\nu$ those of OS; then, since $q \cos\lambda = (y\cos\gamma - z\cos\beta) = p\,(c^{-2}-b^{-2})\,yz$, with similar values for $q\cos\mu$ and for $q\cos\nu$, and since

$$s^{-2} = a^{-2}\cos^2\lambda + b^{-2}\cos^2\mu + c^{-2}\cos^2\nu,$$

therefore, at once,

$$s^2\left[a^{-2}\,(c^{-2}-b^{-2})^2\,y^2z^2 + \&\text{c.}\right] = \left[(c^{-2}-b^{-2})^2\,y^2z^2 + \&\text{c.}\right];$$

hence, multiplying by p^2, substituting for p^2s^2 its constant value $a^2b^2c^2 \div \delta^2$ for every geodesic touching the line of curvature of which δ is the parameter (SALMON's *Geometry of Three Dimensions*, 3rd edit., art. 416), for p^{-2} its known equivalent $a^{-4}x^2 + b^{-4}y^2 + c^{-4}z^2$, and for $a^2b^2c^2$ its equivalent from the equation of the surface $b^2c^2x^2 + c^2a^2y^2 + a^2b^2z^2$, and clearing of fractions, we get, between the coordinates xyz of every point R of the required locus, the homogeneous equation

$$\left[b^4c^4x^2 + c^4a^4y^2 + a^4b^4z^2\right]\left[a^2\,(b^2-c^2)^2\,y^2z^2 + \&\text{c.}\right]$$
$$= \delta^2\left[b^2c^2x^2 + c^2a^2y^2 + a^2b^2z^2\right]\left[a^4\,(b^2-c^2)^2\,y^2z^2 + \&\text{c.}\right],$$

which is that of a sextic cone, passing doubly through the three axes of the ellipsoid, and touching doubly in its three principal planes the quadric cone connecting the line of curvature with the centre of the surface.

For, a complete line of curvature, consisting of course of a twin pair of opposite ovals, the complete locus in question, which with its determining cone is evidently symmetrical with respect to the three principal planes of the surface, is in general doubly tripartite in form; consisting, as the rule, of a twin pair of acnodal points at the extremities of the axis of the surface internal to the cone of connection of the ovals, of a twin pair of dumb-bell lemniscates circumscribed to the two ovals at their two tetrads of vertices, and of a twin pair of undulating reentrants intersecting quadruply at the intersections of the remaining two axes of the surface. The two real cyclic planes passing through the mean axes of the ellipsoid, separating the opposite twin regions of the reentrants, which always pass through the extremities of that axis, from those of the remaining two twin parts of the curve, which always lie in those of the twin ovals of the line.

Putting, successively, $x^2 = a^2$, $y^2 = b^2$, $z^2 = c^2$, in the preceding equation of the determining cone, and taking the quadratic parts only of the resulting equations in the remaining two variables; we get, for the three pairs of directions in which the curve passes doubly through the three pairs of opposite vertices of the ellipsoid, the three equations

$$c^4\,(a^2-b^2)^2\,(b^2-\delta^2)\,y^2 + b^4\,(a^2-c^2)^2\,(c^2-\delta^2)\,z^2 = 0,$$
$$a^4\,(b^2-c^2)^2\,(c^2-\delta^2)\,z^2 + c^4\,(b^2-a^2)^2\,(a^2-\delta^2)\,x^2 = 0,$$
$$b^4\,(c^2-a^2)^2\,(a^2-\delta^2)\,x^2 + a^4\,(c^2-b^2)^2\,(b^2-\delta^2)\,y^2 = 0,$$

which show that the three pairs of parallels to them through the centre are the three corresponding principal sections of the quadric cone whose

equation is $$\frac{(a^2-\delta^2)}{a^4\,(b^2-c^2)^2}\,x^2 + \frac{(b^2-\delta^2)}{b^4\,(c^2-a^2)^2}\,y^2 + \frac{(c^2-\delta^2)}{c^4\,(a^2-b^2)^2}\,z^2 = 0,$$

and which breaks up into a pair of planes, real or imaginary, when $\delta^2 = a^2$

or b^2 or c^2, that is, when the two ovals of the line of curvature come together and coincide with a principal section of the ellipsoid.

Hence, when $\delta^2 = b^2$, in which case alone the two planes of the pair are real, the two lemniscates of the locus circumscribed to the two ovals of the line become two figures of eight, intersecting at right angles the principal ellipse ac at the four real umbilici of the surface, and having their nodal centres at the extremities of its least axis. In the same case, the two reentrants of the locus, while passing through the extremities of the mean axis at the angle of intersection of the two planes, have common tangents parallel to that axis at their passages through the extremities of the greatest axis of the surface.

Making, successively, $b^2 = c^2$, $c^2 = a^2$, $a^2 = b^2$, in the general equation of the determining cone; it assumes, in the three cases respectively, the three simplified forms :—

$$(y^2+z^2)\,x^2\left[a^2(a^2-\delta^2)(y^2+z^2)+b^2(b^2-\delta^2)\,x^2\right]=0,$$
$$(z^2+x^2)\,y^2\left[b^2(b^2-\delta^2)(z^2+x^2)+c^2(c^2-\delta^2)\,y^2\right]=0,$$
$$(x^2+y^2)\,z^2\left[c^2(c^2-\delta^2)(x^2+y^2)+a^2(a^2-\delta^2)\,z^2\right]=0,$$

which show, for a surface of revolution, that the determining cone breaks up into a cylinder of evanescent radius round the polar axis, a pair of parallel planes coinciding with the equatorial plane, and the cone intersecting the surface in the line of curvature; and that the locus itself consists, in consequence, of the two poles of the surface, of the equatorial circle taken doubly, and of the two parallel circles constituting the two ovals of the line; which constitute respectively, for that case, the two acnodal points, the two undulating reentrants, and the two dumb-bell lemniscates, of the general case.

6777. (By Dr. Macfarlane, F.R.S.E.)—Find all the meanings which the term *first cousin* may have in accordance with the English laws of marriage.

Solution by the Proposer.

The general relationship expressed by *first cousin* is denoted by c^{2-2}. Now $\Sigma c^{2-2}\,A = \Sigma c\,(m+f)\,c\,(m+f)\,c^{-1}\,(m+f)\,c^{-1}\,A,$

$$= \Sigma\left\{c\,mc\,mc^{-1}\,mc^{-1}+c\,mc\,mc^{-1}fc^{-1}+c\,mc\,fc^{-1}mc^{-1}+c\,mc\,fc^{-1}fc^{-1}\right.$$
$$\left.+\,c\,fc\,mc^{-1}mc^{-1}+c\,fc\,mc^{-1}fc^{-1}+c\,fc\,fc^{-1}mc^{-1}+c\,fc\,fc^{-1}fc^{-1}\right\}A$$
$$= \Sigma\,(c^{2-2})^1+2\Sigma\,(c^{2-2})^2+3\Sigma\,(c^{2-2})^3+\ldots+8\Sigma\,(c^{2-2})^8\,;$$

where 1 denotes that one and one only of the forms occurs in the person; 2, two and two only, etc.

For the sake of shortness, let the specific relationships be denoted by 1, 2, 3, 4, 5, 6, 7, 8, in order,

$$\Sigma\,(c^{2-2})^1 = \Sigma\left\{1+2+3+4+5+6+7+8\right\},$$
$$\Sigma\,(c^{2-2})^2 = \Sigma\left\{12+13+14+15+16+17+18+23+24+25+26+27+28\right.$$
$$+34+35+36+37+38+45+46+47+48$$
$$\left.+56+57+58+67+68+78\right\}.$$

An examination of the first seven of these terms will suffice to determine all the non-existent terms,

$$12 = c\,mc\,mc^{-1}mc^{-1}\,.\,c\,mc\,mc^{-1}fc^{-1} = c\,mc\,(mc^{-1}mc^{-1}\,.\,mc^{-1}fc^{-1}),$$

(by § 10 of Article). This reduces the question to whether the relationship within the bracket is necessarily non-existent by the laws of marriage. The question is, whether the father of the father of a person can be the father of the mother of the person. By Rule (§ 17 of Article), this is reduced to whether $m\,.\,mc^{-1}c\,fc\,mc^{-1}m$ is non-existent; that is, whether a man can be the husband of a daughter of his father.

13 reduces to $c\left\{mc\,mc^{-1}\,.\,mc\,fc^{-1}\right\}mc^{-1}$, which depends on whether $mc\,mc^{-1}\,.\,mc\,fc^{-1}$ is non-existent. By the Rule referred to above, this is reduced to the question whether $m\,.\,mc^{-1}c\,fc^{-1}m\,cm$ is non-existent; that is, whether a man can be the husband of the mother of his son.

$14 = c\left\{mc\,mc^{-1}mc^{-1}\,.\,mc\,fc^{-1}fc^{-1}\right\}$. The ultimate question is, whether $m\,.\,mc^{-1}c\,fc\,fc^{-1}mc\,mc^{-1}m$ can exist; that is, whether a man can be the husband of a daughter of the wife of his father.

$15 = (c\,mc\,.\,c\,fc)\,mc^{-1}mc^{-1}$. Reduces as in the case of 12.

$16 = c\,mc\,mc^{-1}mc^{-1}\,.\,c\,fc\,mc^{-1}fc^{-1}$. Reduces to

$$m\,.\,mc^{-1}c\,fc\,mc^{-1}fc^{-1}c\,mc\,mc^{-1}m,$$

that is, whether a man can be the husband of a daughter of the father of the wife of a son of his father.

$17 = (c\,mc\,mc^{-1}\,.\,c\,fc\,fc^{-1})\,mc^{-1}$. Reduces to

$$m\,.\,mc^{-1}c\,fc\,fc^{-1}mc\,mc^{-1}m;$$

that is, whether a man can be the husband of a daughter of the wife of his father.

$18 = c\,mc\,mc^{-1}mc^{-1}\,.\,c\,fc\,fc^{-1}fc^{-1}$. Reduces to

$$m\,.\,mc^{-1}c\,fc^{-1}fc^{-1}c\,mc\,mc^{-1}m;$$

that is, whether a man can be the husband of the mother of the wife of a son of his father.

Of these questions only that of 12 and 15 is answered in the negative by the English law.

Hence the terms 12 and 15 are non-existent, and by the symmetry of the law, 34, 56, 78, and 26, 37, 48. The relationships of the third degree are rendered non-existent, only if they contain a non-existent relationship of the second degree. Hence

$$\Sigma\,(c^{2-2})^3 = \Sigma\left\{136+138+146+147+167+168+235+238+245+247 +257+258+358+368+457+467\right\}.$$

And $$\Sigma\,(c^{2-2})^4 = \Sigma\left\{1368+1467+2358+2457\right\}.$$

All relationships of a higher degree are non-existent. Thus a person's first cousins fall into 48 classes; these sub-divided according to sex fall into 96 classes; and if the sex of the person who is the origin of the relationship be taken into account, the word is susceptible of 192 meanings.

6860. (By Professor MATZ, M.A.)—If the curve $\rho = m\sin 2\theta$ be inscribed in the ellipse $\rho^2 = b^2 \div (1-e^2\cos^2\theta)$; prove that

$$3e^2\sin^2\theta = (1-e^2+e^4)^{\frac{1}{2}}+2e^2-1.$$

Solutions by A. McMurchy, B.A.; E. Rutter; *and others.*

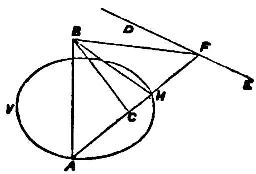
Since ρ and $\frac{d\rho}{d\theta}$ are the same for both curves at their points of contact, we have, by equating values of these quantities,

$$m\sin 2\theta = \frac{b}{(1-e^2\cos^2\theta)^{\frac{1}{2}}}, \text{ and } m = \frac{-e^2 b\sin\theta\cos\theta}{(1-e^2\cos^2\theta)^{\frac{3}{2}}\,.\,2\cos 2\theta};$$

whence $3e^2\sin^2\theta - (4e^2-2)\sin\theta = 1-e^2$, therefore, &c.

7045. (By Dr. Curtis.)—Rays proceeding from any point (or points) on either of the focal conics of a confocal system of surfaces of the second degree are reflected by any of these surfaces; prove that the reflected rays all pass through the same focal conic.

Solution by J. Young, M.A.; *the* Proposer; *and others.*



It may be proved, more generally, that, if rays which touch a surface of the second degree suffer reflection at a confocal surface, they will after reflection touch the same surface again. This appears at once from the fact that the axes of any enveloping cone to a surface of the second degree are the normals through the vertex of the cone to the three confocal surfaces which can be described through this vertex; the areas defined by the focal conics are the surfaces of the system of which one axis vanishes, and, consequently, the theorem is applicable to them. The question proposed may be deduced more immediately, however, from the theorem, that the tangent plane to any surface of the second degree and the normal at its point of contact cut any one of its principal planes in a line and point which are polar and pole with regard to the confocal conic in that plane; for, if we suppose A a point on the focal conic AVH from which rays proceed, B a point on the confocal surface where reflexion takes place, C the point where the normal to the surface cuts the plane of the focal conic, and DE the line in which the tangent plane cuts same, producing AC to cut DE in F and the conic in H, since ACHF is cut harmonically, B . ACHF is a harmonic pencil, and therefore, as $\angle$CBF $= \frac{1}{2}\pi$, $\angle$ABC $=$ HBC, and therefore BH is the reflected ray. [This includes as a particular case, a question given in Parkinson's *Optics*, ed. 1859, p. 262, *Ex.* 18; for the point at infinity on the axis of a paraboloid is situated on each of its confocal parabolæ.]

6886. (By S. Constable, M.A.)—Prove that the distance between the centre of the circumscribing circle of a triangle and the orthocentre is

$$[9R^2-(a^2+b^2+c^2)]^{\frac{1}{2}}.$$

Solution by E. RUTTER; A. McMURCHY; *and others.*

Let O be the centre of the circumscribing circle, and P the orthocentre; then $OA = R$, $AP = 2R\cos A$, and $\angle OAP = C-B$; hence

$$OP^2 = R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C-B)$$
$$= 5R^2 - 4R^2\sin^2 A + 2R^2 - 4R^2\sin^2 B + 2R^2 - 4R^2\sin^2 C = \text{\&c.}$$

7096. (By R. KNOWLES, B.A.)—If m, n are positive integers, prove that

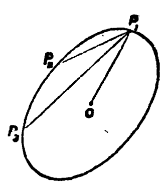
$$\frac{2^n}{m+1} - \frac{n\,.\,2^{n-1}}{(m+1)(m+2)} + \frac{n(n-1)\,2^{n-2}}{(m+1)(m+2)(m+3)} \dots$$
$$\dots + (-1)^n \frac{n(n-1)(n-2)\dots 1}{(m+1)(m+2)\dots(m+n+1)}$$
$$= \frac{1}{m+1} + \frac{n}{m+2} + \frac{n(n-1)}{1\,.\,2}\,.\,\frac{1}{m+3} \dots + \frac{1}{m+n+1}.$$

Solution by Rev. J. L. KITCHIN, M.A.; CHRISTINE LADD, B.A.; *and others.*

If the part under the integral sign of $\int_0^1 x^m (1+x)^n$ be expanded by the binomial theorem, we get the right-hand series; and if we integrate the same by parts, we get the left-hand series; hence the equality.

7022. (By Rev. H. G. DAY, M.A.)—In a closed convex curve, if m be the average area of a triangle one of whose angular points is the centre G, the two others being taken at random in the area; m_1 the average area when all three are taken at random; k_1, k_2 the principal semi-axes of gyration; and S the area; prove that $m_1 = \frac{3}{4}m + \dfrac{k_1^2 k_2^2}{2S}$.

Solution by the PROPOSER; BELLE EASTON; *and others.*



Let $\quad S^2 u = \int_0^{2\pi}\int_0^{\theta} r^3 r_1^3 \sin(\theta-\theta_1)\, d\theta_1\, d\theta,$

CG being the origin. Then, obviously, $m = \frac{2}{9}u$.

Also $v = \iint r_1^4 r^4 \sin^2(\theta-\theta_1)\, d\theta\, d\theta_1 = 32 k_1^2 k_2^2 S^2.$

Now $\quad 48 S^3 m_1 = \iiint K\,[r_3^2 d\theta_3 - r_1 dr_3 \sin(\theta_3-\theta_1)]$

$$\times [r_2^2 d\theta_2 - r_1 d r_2 \sin(\theta_2-\theta)_1]\,[r_1^2 d\theta_1],$$

where $\quad K = r_1 r_2 \sin(\theta_2 - \theta_1) + r_2 r_3 \sin(\theta_3 - \theta_2) - r_1 r_3 \sin(\theta_3 - \theta_1)$,
the integral being taken with the positive sign as long as θ_2 lies between θ_1 and θ_3, and with the negative in other cases.

This, integrated by ordinary rules, gives

$$48 S^3 m_1 = 8S^3 u + \tfrac{1}{4}v, \quad \text{or} \quad m_1 = \tfrac{1}{6}m + \frac{k_1^2 k_2^2}{2S}.$$

6973. (By Rev. T. W. OPENSHAW, M.A.)—On a focal chord of a parabola as diameter is described a circle cutting the parabola again in P, Q; prove that the circle PSQ will touch the parabola.

Solution by E. W. SYMONS, M.A.; R. KNOWLES, B.A.; *and others.*

Let $y - mx + am = 0$ be a focal chord of $y^2 = 4ax$; then a circle on this will take the form $(y - mx + am)(y + mx + d) = \lambda(y^2 - 4ax)$..............(1). The condition for a circle gives $\lambda = 1 + m^2$; and the ordinate of centre found from combining $y - mx + am = 0$ and $y^2 = 4ax$, is $\dfrac{2a}{m} = \dfrac{am + d}{2m^2}$ [found from (1)]; whence $d = 3am$. Again, a circle through P, Q takes the form $(y + mx + 3am)(y - mx + c) = (1 + m^2)(y^2 - 4ax)$; if this passes through $(a, 0)$, we find $c = -\dfrac{a}{m}$; hence the other chord of intersection is $y - mx - m^{-1}a = 0$, which is a tangent to $y^2 = 4ax$ at $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$, *i.e.*, the vertex of the diameter which bisects the focal chord.

[This theorem forms Ex. 941 in Prof. WOLSTENHOLME's *Math. Problems.*]

6690. (By J. HAMMOND, M.A.)—Prove that the limiting value, when $m = 0$, of
$$\frac{1}{\Gamma(m)} \int_0^1 e^{xy} \frac{dy}{(1-y)^{1-m}} = e^x.$$

Solution by D. EDWARDES; J. O'REGAN; *and others.*

Making the substitution $y = \dfrac{z}{1+z}$, we have

$$\int_0^1 e^{xy} \frac{dy}{(1-y)^{1-m}} = \Sigma_{n=0}^{n=\infty} \frac{x^n}{n!} \int_0^\infty \frac{z^n dz}{(1+z)^{1+m+n}} = \Sigma_{n=0}^{n=\infty} \frac{x^n}{n!} \mathrm{B}(n+1, m)$$

(WILLIAMSON's *Integral Calculus*, p. 157

$$= \Sigma_{n=0}^{n=\infty} \frac{x^n}{n!} \frac{\Gamma(n+1)\Gamma(m)}{\Gamma(n+m+1)} = \Sigma_{n=0}^{n=\infty} \frac{x^n}{n!},$$

dividing each term by $\Gamma(m)$ and putting $m = 0$.

7023. (By Rev. W. A. WHITWORTH, M.A.)—Four different rectangular parallelepipeds on square bases have their diagonals equal, and all their edges commensurable with the diagonal; find the solution in lowest integers.

Solution by G. HEPPEL, M.A.; CHARLOTTE A. SCOTT; *and others.*

Let a, a, b, be the edges of one such parallelepiped, and d the diagonal; then $d^2 = 2a^2+b^2$. Let $b = d-e$; then $2de = 2a^2+e^2$; hence e must be even. Let $e = 2f$; then $d = \frac{a^2}{2f}+f$; hence a must be even. Let $a = 2g$, then $d = \frac{2g^2}{f}+f$; hence, taking any value for g, and separating $2g^2$ into all possible pairs of factors f and h, we get

$$d = h+f, \quad a = 2g, \quad b = h-f.$$

For example, suppose $g = 6$, the factors of 72 are 72, 1; 36, 2; 24, 3; 18, 4; 12, 6; 9, 8; and we have $a = 12$; $b = 71, 34, 21, 14, 6, 1$; $d = 73, 38, 27, 22, 18, 17$.

By taking successive values of g, we find that the first instances of d being repeated four times are in the following sets:—

8	31	22	18	21	33	20	17	33	22	11	33
10	49	51	14	47	51	34	17	51	36	3	51
30	69	81	36	63	81	54	27	81	56	17	81

The next instance gives a set of *six*,

14 97 99 24 93 99 44 77 99 54 63 99 60 51 99 66 33 99.

7094. (By D. EDWARDES.)—Prove that the area of a triangle in terms of the radii (ρ_1, ρ_2, ρ_3) of the escribed circles of its orthocentric triangle is

$$\tfrac{1}{2}(\rho_1+\rho_2)(\rho_2+\rho_3)(\rho_3+\rho_1)(\rho_1\rho_2+\rho_2\rho_3+\rho_3\rho_1)^{-\frac{1}{2}}.$$

Solution by E. MORGAN, M.A.; R. KNOWLES, B.A.; *and others.*

If Δ be the area of a triangle, R radius of circumscribed circle, 2σ perimeter of orthocentric triangle; then, by a known theorem, $\Delta = R\sigma$. Now, in any triangle, $r_1+r_2+r_3 = r+4R$,

$$r = r_1r_2r_3 \div (r_1r_2+r_2r_3+r_3r_1), \quad r_1r_2+r_2r_3+r_3r_1 = (\text{semi-perimeter})^2.$$

Hence, for the orthocentric triangle, we have

$$\sigma = (\rho_1\rho_2+\rho_2\rho_3+\rho_3\rho_1)^{\frac{1}{2}},$$

and $2R = \rho_1+\rho_2+\rho_3-\rho_1\rho_2\rho_3 \div \rho_1\rho_2+\rho_2\rho_3+\rho_3\rho_1$

$= (\rho_1+\rho_2)(\rho_2+\rho_3)(\rho_3+\rho_1) \div \rho_1\rho_2+\rho_2\rho_3+\rho_3\rho_1$, therefore, &c.

[This, of course, is the same as proving that the area of the triangle formed by the escribed centres of any triangle is, with the usual notation, $\tfrac{1}{2}(r_1+r_2)(r_2+r_3)(r_3+r_1)(r_1r_2+r_2r_3+r_3r_1)^{-\frac{1}{2}}$].

6791. (By E. W. Symons, M.A.)—A chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches $\frac{x^2}{a^6}+\frac{y^2}{b}=\frac{1}{(a^2-b^2)^2}$; prove that the normals at its extremities meet on the ellipse.

Solution by D. Edwardes; E. Rutter; *and others.*

The normals at the ends of the chords

$$\frac{lx}{a}+\frac{my}{b}=1,\quad \frac{x}{al}+\frac{y}{bm}=-1,$$

meet in a point (Wolstenholme's *Problems*); hence, if one chord be $\frac{ax}{\cos\phi}-\frac{by}{\sin\phi}=a^2-b^2$, the other will be $\frac{x\cos\phi}{a^3}-\frac{y\sin\phi}{b^3}=-\frac{1}{a^2-b^2}$ and the envelope of this is $\quad \frac{x^2}{a^6}+\frac{y^2}{b^6}=\frac{1}{(a^2-b^2)^2}.$

6625. (By C. Leudesdorf, M.A.)—Equilateral triangles BDC, CEA, AFB are drawn externally on the sides of a plane triangle ABC; prove that, if AD, BE be joined, they will intersect at a point whose distance from AB is $\frac{4}{\text{AB}}\cdot\frac{(\text{AFC})(\text{BFC})}{3(\text{ABC})+(\text{BCD})+(\text{CAE})+(\text{ABF})}$, where (AFC) denotes the area of the triangle AFC, &c.

Solution by the Rev. T. R. Terry, M.A., F.R.A.S.; E. Rutter; *and others.*

If $l=(\text{CAF})=(\text{BAE})$, &c., then $(\text{BCD})=m+n-\Delta$. Equation to AD is $my=nz$, and to BE is $lx=nz$. Hence AD, BE, CF meet in the point where $lx=my=nz\equiv\frac{\Delta lmn}{lm+mn+nl}$; therefore perpendicular from the point on AB $=\frac{2\Delta}{\text{AB}}\cdot\frac{lm}{lm+mn+nl}$. To throw this result into the form given in question, we must prove that

$$2(lm+mn+nl)=\Delta(l+m+n)\text{........................}(1).$$

Now, since $m+n-\Delta=pa^2$, &c., where $p=\frac{1}{4}\sqrt{3}$, $2l=\Delta+p(b^2+c^2-a^2)$, therefore $4(lm+mn+nl)=2\Delta[3\Delta+p(a^2+b^2+c^2)]=2\Delta(l+m+n)$, the required result.

6644. (By W. J. C. Sharp, M.A.)—When the catalecticant of a binary $2n$-ic $(a, b, c \dots k, l)(x, y)^{2n}$ vanishes, prove that the n quantities to the sum of whose $2n^{\text{th}}$ powers the quantic is reducible are the factors of the canonizant of $(a, b, c \dots k)(x, y)^{2n-1}$.

Solution by the PROPOSER.

Let p_1x+q_1y, p_2x+q_2y, &c. be the n quantities required,

$$\therefore\ (a, b, c \ldots k, l)(x, y)^{2n} \equiv (p_1x+q_1y)^{2n} + (p_2x+q_2y)^{2n} + \&c.,$$

$$\therefore\ ax+by \equiv p_1^{2n-1}(p_1x+q_1y)+p_2^{2n-1}(p_2x+q_2y)+\&c.,$$

$$bx+cy \equiv p_1^{2n-2}q_1(p_1x+q_1y)+p_2^{2n-2}q_2(p_2x+q_2y)+\&c.,$$

$$\&c. \qquad \&c. \qquad \&c.$$

Hence, by substitution,

$$\left|\begin{array}{ccc} ax+by\,. & bx+cy\,. & cx+dy \ldots \\ bx+cy\,. & cx+dy\,. & dx+by \ldots \\ \&c. & \&c. & \&c. \\ \ldots & \ldots \ldots & \ldots \ldots \end{array}\right| \text{vanishes if } p_1x+q_1y=0, \text{ or if } p_2x+q_2y=0, \&c.$$

and therefore p_1x+q_1y, &c. are the factors, &c.

6769. (By Professor TOWNSEND, F.R.S.)—Three forces in a common space being supposed transferred to the centre of the quadric determined by their three lines of direction in the space; show, on elementary principles, that

(*a*) The plane of the resultant moment is conjugate to the direction of the resultant force with respect to the surface;

(*b*) The pitch of the equivalent wrench $= abc \div r^2$; where a, b, c are the semi-axes of the surface, and r its radius in the direction of the force.

Solution by the PROPOSER; T. WOODCOCK, B.A.; *and others.*

Of these properties, which are true generally for any number of forces acting along generators of the same system of any ruled quadric in their space, the first, which has already been given separately in Quest. 6535, is at once manifest from the consideration that as the forces, whatever be their number, have no moments round the two generators E and F of the opposite system at the two extremities P and Q of the diameter POQ, which coincides in direction with the resultant of their transference to the centre O of the surface, and as the planes of their moments at P and Q are parallel to each other and to that of their moment at O, the latter is in consequence parallel to the two tangent planes to the surface at P and Q, and is therefore the diametral plane conjugate to the direction PQ; and the second may be readily proved from it as follows.

Denoting by x, y, z the three coordinates to the three principal planes of the surface of either extremity P of the diameter PQ, by r and p the radius OP and the perpendicular from O on the tangent plane at P, by a, b, c the three semi-axes of the surface, by X, Y, Z the three components parallel to the axes of the resultant R of the system of forces transferred to O, and by L, M, N those round the same of the corresponding moment G resulting from the transference; then, since, by the above,

$$\mathrm{L} : \mathrm{M} : \mathrm{N} = \frac{\mathrm{X}}{a^2} : \frac{\mathrm{Y}}{b^2} : \frac{\mathrm{Z}}{c^2} = \frac{x}{a^2}\cdot\frac{\mathrm{R}}{r} : \frac{y}{b^2}\cdot\frac{\mathrm{R}}{r} : \frac{z}{c^2}\cdot\frac{\mathrm{R}}{r},$$

therefore $\mathrm{L} = k\cdot\dfrac{\mathrm{X}}{a^2} = k\cdot\dfrac{\mathrm{R}}{r}\cdot\dfrac{x}{a^2},\quad \mathrm{M} = k\cdot\dfrac{\mathrm{R}}{r}\cdot\dfrac{y}{b^2},\quad \mathrm{N} = k\cdot\dfrac{\mathrm{R}}{r}\cdot\dfrac{z}{c^2},$

where k remains to be determined; hence

$$G = [L^2+M^2+N^2]^{\frac{1}{2}} = k \cdot \frac{R}{r} \cdot \left[\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right]^{\frac{1}{2}} = \frac{k}{rp} R,$$

and therefore $\dfrac{G}{R} = \dfrac{k}{rp}$, and $\dfrac{G}{R} \cdot \dfrac{p}{r}$ = pitch of wrench $= \dfrac{k}{r^2}$;

and it remains only to show that the value of k is that given above.

Conceiving the entire system of forces transformed, as they evidently may, into two pairs of forces P and P′, Q and Q′ acting along the two pairs of generators E and E′, F and F′ of their system passing through the two pairs of opposite extremities A and A′, B and B′ of the two real axes AOA′ and BOB′ of the surface; we have then, evidently,

$$X = (Q \mp Q') \frac{a}{(a^2+c^2)^{\frac{1}{2}}}, \quad Y = (P \mp P') \frac{b}{(b^2+c^2)^{\frac{1}{2}}},$$

$$Z = (P \pm P') \frac{c}{(a^2+c^2)^{\frac{1}{2}}} + (Q \pm Q') \frac{c}{(b^2+c^2)^{\frac{1}{2}}}, \quad L = (Q \mp Q') \frac{bc}{(a^2+c^2)^{\frac{1}{2}}},$$

$$M = (P \mp P') \frac{ac}{(b^2+c^2)^{\frac{1}{2}}}, \quad N = (P \pm P') \frac{ab}{(a^2+c^2)^{\frac{1}{2}}} + (Q \pm Q') \frac{ab}{(b^2+c^2)^{\frac{1}{2}}},$$

and consequently

$$\frac{L}{X} = \frac{bc}{a} = \frac{abc}{a^2}, \quad \frac{M}{Y} = \frac{ac}{b} = \frac{abc}{b^2}, \quad \frac{N}{Z} = \frac{ab}{c} = \frac{abc}{c^2};$$

hence $k = abc$, and therefore, &c.

6962. (By J. J. Walker, M.A.)—Show that the cosine of half the angle between tangents drawn from a point on an ellipse to an inner confocal ellipse, varies inversely as the diameter of the former parallel to its tangent at that point.

7160. (By J. Griffiths, M.A.)—If $2a$ be the angle between the pair of tangents from an external point (x', y') to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b} = 1$, and a_1, b_1 the semi-axes of the confocal ellipse through (x', y'), show that

$$\sin^2 a = \frac{a_1^2 a^2 y'^2 + b_1^2 b^2 x'^2}{a_1^4 y'^2 + b_1^4 x'^2}.$$

Solution by J. J. Walker, M.A.; W. M. Mee, B.A.; *and others.*

We have $\tan^2 2a = \dfrac{4(b^2x'^2+a^2y'^2-a^2b^2)}{[x'^2+y'^2-(a^2+b^2)]^2}$, or, if $k^2 = a_1^2-a^2 = b_1^2-b^2$, r_1 = semi-diameter of outer ellipse parallel to its tangent at $x'y'$,

$$\tan^2 2a = \frac{4k^2(r_1^2-k^2)}{(2k^2-r_1^2)^2};$$

whence $\qquad (2\cos^2 a - 1)^2 = \dfrac{1}{1+\tan^2 2a} = \dfrac{(2k^2-r_1^2)^2}{r_1^4}$, $\cos a = \dfrac{k}{r_1}$.

Again, $\sin^2 a = 1-\cos^2 a = \dfrac{r_1^2-k^2}{r_1^2} = \dfrac{a_1^4y'^2+b_1^4x'^2-k^2a_1^2b_1^2}{a_1^4y'^2+b_1^4x'^2}$

$$= \frac{a_1^4y'^2+b_1^4x'^2-(a_1^2-a^2)\,a_1^2y'^2-(b_1^2-b^2)\,b_1^2x'^2}{a_1^4y'^2+b_1^4x'^2} = \frac{a_1^2a^2y'^2+b_1^2b^2x'^2}{a_1^4y'^2+b_1^4x'^2}.$$

6623. (By Professor HUDSON, M.A.)—If the angular radius of the small circle drawn round a triangle be $\tan^{-1} 2$, the area of the triangle be one-sixth of the sphere, and one side be a quadrant, find all the parts of the triangle.

Solution by W. M. COATES; CHARLOTTE A. SCOTT; *and others.*

We have $\tan R = \dfrac{\tan \frac{1}{2}a}{\cos(S-A)}$, since $a = \frac{1}{2}\pi$ and $R = \tan^{-1} 2$;

therefore $2 = \sec(S-A)$, therefore $S-A = \frac{1}{3}\pi$ or $B+C-A = \frac{2}{3}\pi$.
But $A+B+C-\pi = \frac{2}{3}\pi$, therefore $A = \frac{1}{2}\pi$,
and we have two equations to determine B and C, namely $B+C = \frac{7}{6}\pi$, $\cos A = 0 = \cot B \cot C$, therefore either B or $C = \frac{1}{2}\pi$. If $B = \frac{1}{2}\pi$, we have

$$a = \tfrac{1}{2}\pi,\quad b = \tfrac{1}{2}\pi,\quad c = \tfrac{2}{3}\pi,\quad A = \tfrac{1}{2}\pi,\quad B = \tfrac{1}{2}\pi,\quad C = \tfrac{2}{3}\pi.$$

The triangle is therefore that included on a globe between the equator and the meridians of 0° and 120°.

6159. (By Professor COCHEZ.)—Un paraboloïde elliptique est coupé par un plan perpendiculaire à l'axe. Trouver le plus grand parallelepipède rectangle que l'on puisse inscrire dans la portion de paraboloïde déterminée par le plan sécant.

Solution by Rev. J. L. KITCHIN, M.A.; SARAH MARKS; *and others.*

Let $\dfrac{x^2}{a}+\dfrac{y^2}{b} = 2z$ be the paraboloid, $z = h$ the cutting plane, and the four angular points in the parallel plane, the angular points of a rectangle inscribed in an ellipse, be $(x_1, y_1, z_1)(x_1, -y_1, z_1)$, $(-x_1, y_1, z_1)(-x_1, -y_1, z_1)$. Then the volume is $4x_1y_1(h-z_1)$;

therefore $$x_1y_1\left\{h-\tfrac{1}{2}\left(\frac{x_1^2}{a}+\frac{y_1^2}{b}\right)\right\} = \text{a maximum},$$

$$\therefore\ y_1\left\{h-\tfrac{1}{2}\left(\frac{x_1^2}{a}+\frac{y_1^2}{b}\right)\right\}-\frac{x_1^2y_1}{a}=0,\quad x_1\left\{h-\tfrac{1}{2}\left(\frac{x_1^2}{a}+\frac{y_1^2}{b}\right)\right\}-\frac{x_1y_1^2}{b} = 0;$$

therefore $\dfrac{x_1^3y_1}{a}-\dfrac{y_1^3x_1}{b} = 0$, therefore $x_1=0$, $y_1=0$, which gives an evident minimum, or $\dfrac{x_1^2}{a}-\dfrac{y_1^2}{b} = 0$; therefore $\dfrac{x_1^2}{a} = \dfrac{y_1^2}{b} = \dfrac{h}{2} = z_1$, and the volume $= h^2(ab)^{\frac{1}{2}}$.

7141. (By R. Tucker, M.A.)—Any fourth point (P) is taken on the circumference of the circle ABC; prove that the mid-points of PA, PB, PC form a triangle, similar to ABC and of one quarter its area, and such that its circumscribing circle always touches ABCP at P. [Further results may be obtained by substituting any one of the A, B, C points for P.]

Solution by Rev. T. R. Terry, M.A.; J. O'Regan; *and others.*

If a be the middle point of PA, and PT be the tangent at P to the circle ABC, then clearly ab is parallel to AB and half of it, therefore Δabc is similar to ΔABC and one quarter its area. Again, since $\angle$ APB $= \angle$ ACB $= \angle\, acb$, therefore a circle goes through abcP; and since $\angle$ APT $= \angle$ ABP $= \angle\, ab$P, the circles ABCP, abcP touch at P.

7133. (By Rev. G. Richardson, M.A.)—If AOD, BOE, COF be the perpendiculars from A, B, C on the opposite sides of the triangle ABC, and G be the centroid of the triangle; prove that the circles described about the triangles ADG, BEG, CFG all meet again in a point which is the intersection of OG with the radical axis of the nine-point circle, and circumscribing circle of ABC.

Solution by Prof. Genese, M.A.; M. Jenkins, M.A.; *and others.*

1. Since AO . OD = BO . OE = CO . OF, the line GO meets the three circles in the same point,—K say.

2. GO . OK = AO . OD = $\frac{1}{2}$ ($\text{R}^2 - \text{SO}^2$). S being the centre of the circle ABC, therefore OK $= \frac{3}{4} \cdot \dfrac{\text{R}^2 - \text{SO}^2}{\text{SO}}$. Let N be the centre of the N . P . C, M the middle point of SN, then MO $= \frac{3}{4}$ OS, ∴ MK $= \frac{3}{4} \dfrac{\text{R}^2}{\text{SO}}$, or 2SN . MK $\frac{3}{4}\text{R}^2 = \text{R}^2 - (\frac{1}{2}\text{R})^2$, therefore K is on the radical axis of circle ABC and N . P . C.

6842. (By G. F. Walker, M.A.)—A system of $2n$ convex lenses, of equal numerical focal length f are placed with their axes in the same straight line and their centres at a distance $4f$ apart, except the two middle ones, which are at a distance $8f$ apart; show that the focal length of a lens which must be placed midway between the two middle ones in order that the image of a bright point at a distance $4f$ in front of the first lens may be formed at an equal distance behind the last lens, is $\dfrac{2(n+1)}{2n+1} f$.

Solution by G. M. REEVES, M.A.; BELLE EASTON; *and others.*

Let $v_1, v_2, v_3 \dots v_n$ be the distances from each of the n lenses, on one side, of the images after refraction through that lens; then we have the n equations

$$\frac{1}{v_1}-\frac{1}{4f}=-\frac{1}{f},\quad \frac{1}{v_2}-\frac{1}{v_1+4f}=-\frac{1}{f},\quad \dots \quad \frac{1}{v_n}-\frac{1}{v_{n-1}+4f}=-\frac{1}{f},$$

whence $$\frac{1}{v_1}=-\frac{3}{4f},\quad \frac{1}{v_2}=-\frac{5}{8f},\quad \frac{1}{v_3}=-\frac{7}{12f};$$

and, by mathematical induction, we can prove that $\frac{1}{v_n}=-\frac{2n+1}{4nf}$. Let F be the numerical focal length of the required lens; then, on account of the symmetrical arrangement of the $2n$ lenses,

$$-\frac{1}{v_n+4f}-\frac{1}{v_n+4f}=-\frac{1}{F},\quad \text{or}\quad \frac{1}{F}=\frac{2}{v_n+4f}=\frac{2}{4f-\frac{4nf}{2n+1}}=\frac{2n+1}{2(n+1)f},$$

therefore $$F=\frac{2(n+1)}{2n+1}f.$$

6734. (By H. G. DAWSON.)—If $ax^4+4bx^3+6cx^2+4dx+e=0$ have a double root α, prove that $H=b^2-ac$, $I=ae-4bd+3c^2$,

$$J=ace+2bcd-ad^2-eb^2-c^3,\quad \alpha=-\frac{b}{a}+\left(\frac{H}{a^2}\pm\frac{3J}{aI}\right)^{\frac{1}{2}}.$$

Solution by SARAH MARKS; G. EASTWOOD, M.A.; *and others.*

Transforming $ax^4+4bx^3+Ic=0$ by substituting $y+\alpha=x$, the equation becomes $$Ay^4+4By^3+6Cy^2=0,$$

$$\frac{H}{a^2}=\frac{B^2-AC}{A^2},\quad \frac{I}{a^2}=\frac{3C^2}{A^2},\quad \text{therefore}\quad \left(\frac{B}{A}\right)^2=\frac{H}{a^2}+\left(\frac{I}{3a^2}\right),$$

therefore $$\left(\frac{B}{A}\right)^2=\frac{H}{a}\pm\frac{3J}{aI};$$

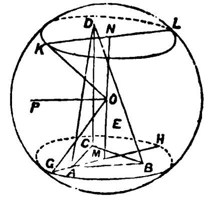
but $$\beta-\alpha+\gamma-\alpha=-\frac{4B}{A},\quad \text{therefore}\quad -\frac{4b}{a}-4\alpha=-\frac{4B}{A},$$

therefore $$\alpha=-\frac{b}{a}+\frac{B}{A},\quad \text{therefore, \&c.}$$

6164. (By Professor SEITZ, M.A.)—If A, B, C, D, E are five random points within a sphere, prove that the chance that any one of the points is within the tetrahedron having the other four points for its vertices is $\frac{9}{143}$.

6169. (By ELIZABETH BLACKWOOD.)—If A, B, C are random points within a sphere of which O is the centre and r the radius, prove that the average volume of the tetrahedron ABCO is $\frac{9}{1024}\pi r^3$.

Solution by Prof. Seitz, M.A.; Prof. Matz, M.A.; *and others.*



Let GH be the diameter of the section of the sphere made by the plane through A, B, C, M its centre, GL the diameter of the section through D parallel to the plane ABC, N its centre, O the centre of the sphere, OP a line such that AB is parallel to the plane MOP.

Now, the required chance is evidently equal to five times the chance that E is within the tetrahedron ABCD.

Let $OG = r$, $\angle GOM = \theta$, and $\angle KON = \mu$: then, giving D all positions within the sphere, we have, for the sum of all the tetrahedrons ABCD,

$$\int_{\theta}^{\pi-\theta} \text{area ABC} \times \tfrac{1}{3}(\cos\mu + \cos\theta)\,\pi r^4 \sin^3\mu\, d\mu$$

$$+\int_0^{\theta} \text{area ABC}\tfrac{1}{3} \times (\cos\mu - \cos\theta)\,\pi r^4 \sin^3\mu\, d\mu$$
$$= \tfrac{1}{18}\pi r^4 (8 - 4\sin^2\theta - \sin^4\theta)\,\text{area ABC}\,.........(1).$$

Now, let $MA = x$, $AB = y$, $AC = z$, $\angle BAM = \phi$, $\angle CAM = \psi$, $\angle MOP = \omega$, and the angle the plane MOP makes with a fixed plane through OP $= \rho$. Then $(1) = \frac{1}{36}(8 - 4\sin^2\theta - \sin^4\theta)\,\pi r^4 yz \sin(\phi+\psi) = f(\theta)\, yz\sin(\phi+\psi)$. An element of the sphere at A is $r\sin\theta\, d\theta\,.\,2\pi x\, dx$, at B it is $y^2 dy\, d\phi\, d\omega$, and at C it is $\sin(\phi+\psi)\sin\omega\, z^2 dz\, d\psi\, d\rho$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of x, 0 and $r\sin\theta = x$, and tripled; of ϕ, $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ψ, $-\phi$ and $\frac{1}{2}\pi$, and doubled; of ω, 0 and π; of ρ, 0 and 2π; of y, 0 and $2x\cos\phi = y'$; and of z, 0 and $2x\cos\psi = z'$. Hence, since the whole number of ways the five points can be taken is $(\frac{4}{3}\pi r^3)^5$, the chance that E is within the tetrahedron ABCD is

$$\frac{6}{(\frac{4}{3}\pi r^3)^5}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{y'}\int_0^{z'} f(\theta)\, r\sin\theta\, d\theta\,.\,2\pi x\, dx\, d\phi\, \sin^2(\phi+\psi)\, d\psi \times \sin\omega\, d\omega\, d\rho\, y^3\, dy\, z^3\, dz$$

$$= \frac{729}{16\pi^4 r^{14}}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi} f(\theta)\sin\theta\, d\theta\, x^9\, dx\, \cos^4\phi\, d\phi\, \sin^2(\phi+\psi)\cos^4\psi\, d\psi\, \sin\omega\, d\omega\, d\rho$$

$$= \frac{729}{4\pi^3 r^{14}}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi} f(\theta)\sin\theta\, d\theta\, x^9\, dx\, \cos^4\phi\, d\phi\, \sin^2(\phi+\psi)\cos^4\psi\, d\psi$$

$$= \frac{243}{64\pi^3 r^{14}}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(\theta)\left[3(\tfrac{1}{2}\pi+\phi)(5-4\cos^2\phi) + 15\sin\phi\cos\phi - 2\sin\phi\cos^3\phi\right]\sin\theta\, d\theta\, x^9\, dx\, \cos^4\phi\, d\phi$$

$$= \frac{405}{4096 r^{10}}\int_0^{\frac{1}{2}\pi}\int_0^{x'}(8 - 4\sin^2\theta - \sin^4\theta)\sin\theta\, d\theta\, x^9\, dx\ .$$

$$= \frac{81}{8192}\int_0^{\frac{1}{2}\pi}(8 - 4\sin^2\theta - \sin^4\theta)\sin^{11}\theta\, d\theta = \frac{9}{715}.$$

Hence the required chance is $\frac{9}{143}$.

The volume of the tetrahedron ABCO $= \frac{1}{6} ryz \cos\theta \sin(\phi+\psi)$; hence the average volume is

$$\frac{6}{(\frac{4}{3}\pi r^3)^3}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{y'}\int_0^{z'} \tfrac{1}{6}r^2\sin\theta\cos\theta\, d\theta\,.\,2\pi x\,dx\,d\phi\,\sin^2(\phi+\psi)d\psi \times \sin\omega\,d\omega\,d\rho\,y^3\,dy\,z^3\,dz$$

$$= \frac{135\pi}{128r^7}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\sin\theta\cos\theta\,d\theta\,x^9\,dx = \frac{27\pi r^3}{256}\int_0^{\frac{1}{2}\pi}\sin^{11}\theta\cos\theta\,d\theta = \frac{9}{1024}\pi r^3.$$

7025. (By W. A. Pick.)—Prove that, if a number of 3 or 4 figures is divisible by 7, half of the tens and units added to the other figure or figures will be divisible by 7.

Solution by C. Hammond; G. Eastwood, M.A.; *and others.*

The property here stated is true for *any* multiple of 7. In fact, if $\frac{1}{2}$N be expressed by $\frac{a+10b}{2}+50c+500d+5000e+\ldots$, and if we take from $\frac{1}{2}$N the number $49c+490d+4900e+\ldots$, which is a multiple of 7, we have for remainder $\frac{a+10b}{2}+c+10d+100e+\ldots$, which accordingly is a multiple of 7 if $\frac{1}{2}$N be so, *i.e.*, if N be so.

6824. (By Byomakesa Chakravarti.)—An infinite number of radii vectores are drawn from the centre of an ellipse of axes $2a$, $2b$; if m be the mean value of the squares of these radii, when the differences of their successive excentric angles are equal, and m' the mean value of the squares when the successive angles between the radii themselves are equal, prove that $m+m' : m-m' = (a+b)^2 : (a-b)^2$.

Solution by J. Hammond, M.A.; D. Edwardes; *and others.*

$$m = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}(a^2\cos^2\phi+b^2\sin^2\phi)\,d\phi = \tfrac{1}{2}(a^2+b^2),$$

$$m = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}} = ab;\ \text{therefore, \&c.}$$

7069. (By D. Edwardes.)—If x, y, z be the distances of a point P from the angular points of a triangle, prove that the mean value of $x^2 \sin 2A + y^2 \sin 2B + z^2 \sin 2C$, as P ranges over the circle about ABC, is three times the area of the triangle.

Solution by W. H. Blythe, M.A.; W. M. Coates, B.A.; *and others.*

The expression given has by a former question been shown $= 4(R^2+d^2) \sin A \sin B \sin C$, where d is the distance of ρ from the centre of the circle round ABC. The mean value required

$$= \int_0^R 4(R^2+x^2) \sin A \sin B \sin C \,.\, 2\pi x \,.\, dx \div \int_0^R . \,2\pi x \,.\, dx$$
$$= 6R^2 \sin A \sin B \sin C.$$

Putting $R \sin A = \frac{1}{2}a$, $R \sin B = \frac{1}{2}b$, this evidently reduces to three times the area of the triangle.

7093. (By H. L. Orchard, M.A.)—If C, S, respectively, be the centre and one focus of the ellipse $3(x^2-a^2)+4y^2=0$, and P be a point on the curve such that the $\angle CSP = \theta$; find (1) the area of the triangle CSP, and (2) for what value of θ this will be a maximum.

Solution by Rev. J. L. Kitchin, M.A.; C. Morgan, B.A.; *and others.*

The semi-axes of the ellipse are a and $\frac{1}{2}a\sqrt{3}$, therefore $e = \frac{1}{2}$; also area of CSP $= \frac{1}{2}ae \,.\, \text{SP} \,.\, \sin\theta = \frac{1}{2}ae\,\dfrac{a(1-e^2)}{1-e\cos\theta}\sin\theta = \frac{3}{16}a^2 \,.\, \dfrac{\sin\theta}{1-\frac{1}{2}\cos\theta}$. For a maximum the difficulty with respect to θ vanishes, whence we obtain $\theta = 60°$.

6975. (By E. Rutter.)—If s_1 = sum of odd terms, and s_2 = sum of even terms, of $(a+b)^n$, show that $s_1^2 - s_2^2 = (a^2-b^2)^n$.

Solution by H. L. Orchard, M.A; J. L. Jenkins, M.A.; *and others.*

$$s_1^2 - s_2^2 = (s_1+s_2)(s_1-s_2) = (a+b)^n (a-b)^n = (a^2-b^2)^n.$$

6968. (By D. Edwardes.)—Prove that

$$(y+z-2x')(z+x-2y')(x+y-2z') - (y+z-2x')(x+x'-y'-z')^2 - \ldots - \ldots$$
$$\equiv 2(x+x'-y'-z')(y+y'-z'-x')(z+z'-x'-y').$$

Solution by G. HEPPEL, M.A.; SARAH MARKS; *and others.*

Putting $u = x+x'-y'-z'$; $v =$ &c.; $w =$ &c.; the identity becomes

$$(v+w)(w+u)(u+v)-(v+w)u^2-(w+u)v^2-(u+v)w^2 = 2uvw,$$

which is obviously true.

7021. (By W. NICHOLLS, B.A.)—Let ABCD be any quadrilateral circumscribable by a circle, P the intersection of the sides AD, BC; L the middle point of CD; and M the point where PL produced meets AB: prove that $\qquad \text{MA} : \text{MB} = \text{AP}^2 : \text{PB}^2.$

Solution by J. S. JENKINS; J. O'REGAN; *and others.*

$$\frac{\text{MA}}{\text{MB}} = \frac{\text{MA}}{\text{MP}}\cdot\frac{\text{MP}}{\text{MB}} = \frac{\sin \text{LPD}}{\sin \text{A}}\cdot\frac{\sin \text{B}}{\sin \text{LPC}} = \frac{\sin \text{LPD}}{\sin \text{L}}\cdot\frac{\sin \text{L}}{\sin \text{LPC}}\cdot\frac{\sin \text{B}}{\sin \text{A}}$$

$$= \frac{\text{LD}}{\text{PD}}\cdot\frac{\text{PC}}{\text{LC}}\cdot\frac{\sin \text{D}}{\sin \text{C}} = \frac{\text{PC}}{\text{PD}}\cdot\frac{\text{PC}}{\text{PD}}.$$

5615. (By S. TEBAY, B.A.)—"In the arbelon of the ancient geometers, three circular segments are drawn on the same straight line (length 6), and on given parts of it (4 and 2)." Given the sum of the three arcs, find the radii when the sum of the areas of the segments is a maximum.

Solution by the PROPOSER; A. MARTIN, M.A.; *and others.*

Let $2a_1$, $2a_2$, ... be the chords, $2\theta_1$, $2\theta_2$, ... the angles which they subtend at the centres, r_1, r_2, ... the radii, and $2s$ the sum of the arcs. Then

$$s = r_1\theta_1 + r_2\theta_2 + \ldots = a_1\theta_1 \operatorname{cosec}\theta_1 + a_2\theta_2 \operatorname{cosec}\theta_2 + \ldots,$$

$$r_1^2\theta_1 + r_2^2\theta_2 + \ldots - (a_1 r_1 \cos\theta_1 + a_2 r_2 \cos\theta_2 + \ldots)$$

$$= a_1^2\theta_1 \operatorname{cosec}^2\theta_1 + a_2^2\theta_2 \operatorname{cosec}^2\theta_2 + \ldots - (a_1^2 \cot\theta_1 + a_2^2 \cot\theta_2 + \ldots),$$

a maximum. Differentiating, we have

$$a_1 \operatorname{cosec}\theta_1 (1-\theta_1\cot\theta_1)\,\delta\theta_1 + a_2 \operatorname{cosec}\theta_2 (1-\theta_2\cot\theta_2)\,\delta\theta_2 + \ldots = 0,$$

$$a_1^2 \operatorname{cosec}^2\theta_1 (1-\theta_1\cot\theta_1)\,\delta\theta_1 + a_2^2 \operatorname{cosec}^2\theta_2 (1-\theta_2\cot\theta_2)\,\delta\theta_2 + \ldots = 0.$$

Multiply the first by λ, and add; then, putting the coefficient of $\delta\theta_1$, $\delta\theta_2$, ... $= 0$, we have $\lambda + a_1 \operatorname{cosec}\theta_1 = 0$, $\lambda + a_2 \operatorname{cosec}\theta_2 = 0$, &c.,

or $$-\lambda = r_1 = r_2 = \text{\&c.} = r, \text{ suppose};$$

therefore $$\frac{s}{r} = \theta_1 + \theta_2 + \ldots = \sin^{-1}\frac{a_1}{r} + \sin^{-1}\frac{a_2}{r} + \ldots;$$

which determines r. In the particular case,

$$\frac{6}{r} = \sin^{-1}\frac{1}{r} + \sin^{-1}\frac{2}{r} + \sin^{-1}\frac{3}{r};$$

from which we find $r = 23\cdot005$, nearly.

6761. (By G. F. Walker, M.A.)—A railway engine is drawing a train of equal carriages connected by spring couplings of strength μ, and the driving power is so adjusted that the velocity is $a + b\sin nt$. Show that, if n be nearly equal to $b\left\{\frac{2\mu}{(M+4m)b^2+4m\kappa^2}\right\}^{\frac{1}{2}}$, the couplings will probably break, M being the mass of a carriage, which is supported on four equal wheels of mass m, radius b, and radius of gyration κ.

Solution by the Proposer; E. Rutter; *and others.*

Let y_r be the distance of the centre of the r^{th} carriage from some fixed point on the line.

Then, the coupling chains being supposed inextensible, and their mass being neglected, the equation of motion of the r^{th} carriage is found to be

$$M\ddot{y}_r + 4m\ddot{y}_r + 4m\kappa^2\frac{\ddot{y}_r}{b} = \frac{y_{r+1} - 2y_r + y_{r-1}}{2}\mu,$$

or
$$\frac{4}{n_1^2}\ddot{y}_r = y_{r+1} - 2y_r + y_{r-1},$$

where n_1 has the value given in the question. We shall now find the period of the small oscillation of the strain without the engine.

The first and last equations of motion

$$\frac{4}{n_1^2}\ddot{y}_1 = y_2 - y_1, \qquad \frac{4}{n_1^2}\ddot{y}_s = -(y_s - y_{s-1}),$$

if s be the number of carriages.

Assume $y_r = z_r \sin pt$, and we find

$$z_{r+1} - z\left(1 - \frac{2p^2}{n_1^2}\right)z_r + z_{r-1} = 0, \qquad z_2 - \left(1 - \frac{4p^2}{n_1^2}\right)z_1 = 0,$$
$$z_s\left(1 - \frac{4p^2}{n_1^2}\right) - z_{s-1} = 0.$$

Let α_1, α_2 be the roots of the equation $x^2 - 2\left(1 - \frac{2p^2}{n_1^2}\right)x + 1 = 0.$

Then
$$z_r = A\alpha_1^r + B\alpha_2^r \text{ and } \alpha_1 + \alpha_2 = 2\left(1 - \frac{2p^2}{n_1^2}\right), \ \alpha_1\alpha_2 = 1,$$
$$A\alpha_1^2 + B\alpha_2^2 - (\alpha_1 + \alpha_2 - 1)(A\alpha_1 + B\alpha_2) = 0,$$
$$(A\alpha_1^s + B\alpha_2^s)(\alpha_1 + \alpha_2 - 1) - (A\alpha_1^{s-1} + B\alpha_2^{s-1}) = 0$$

or, remembering $\alpha_1\alpha_2 = 1$, $\quad A(\alpha_1 - 1) + B(\alpha_2 - 1) = 0,$

and $\quad A\alpha_1^s(\alpha_1 - 1) + B\alpha_2^s(\alpha_2 - 1) = 0, \quad \therefore \alpha_1^s = \alpha_2^s.$

But, if $1-\dfrac{2p^2}{n_1^2}$ be numerically greater than unity, the roots a_1 and a_2 are real and different; hence it must be less than unity, $=\cos\theta$ suppose,

$$1-\frac{2p^2}{n_1^2}=\cos\theta,\quad\text{therefore } p=n_1\sin\frac{\theta}{2}.$$

Also $a_1=\cos\theta+\sqrt{-1}\sin\theta$, $a_2=\cos\theta-\sqrt{-1}\sin\theta$, and if $a_1^s=a_2^s$ we have $\sin s\theta=0$, therefore $s\theta=\kappa\pi$, $\theta=\dfrac{\kappa\pi}{s}$, therefore $p=n_1\sin\dfrac{\kappa\pi}{2s}$.

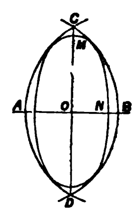
This therefore gives the period of the natural oscillation, and if the period of the driving power nearly coincide with any one of these, large relative motion will probably ensue, and the couplings probably break.

κ has here an integral value, and $p=n_1$ or $\kappa=s$ is the only one which gives a result independent of the number of carriages.

Only the oscillatory part of the motion is above taken into account, and the b in $a+b\sin nt$, which had better be written $a+\beta\sin nt$, has no connection with the radius *of the wheels*.

6604. (By Professor MATZ, M.A.)—Prove that the mean area of the maximum elliptic disc placed upon the surface common to two equal circles of radius unity, that are thrown upon each other at random, is $\frac{1}{8}\pi(3\pi-8)$.

Solution by Professor SEITZ, M.A.



Let A and B be the centres of the two circles, CD the common chord of the circles, and OM, ON the semi-axes of the maximum elliptic disc. Let OM $=a$, ON $=b$, and AB $=z$.

Then the equation to the circle A, referred to AB, CD, is

$$(x+\tfrac{1}{2}z)^2+y^2=1\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

and the equation to the ellipse is

$$a^2x^2+b^2y^2=a^2b^2\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) and (2), by eliminating y, we have

$$(a^2-b^2)\,x^2-b^2zx+b^2\,(1-a^2-\tfrac{1}{4}z^2)=0\;\ldots\ldots\ldots\ldots\ldots(3).$$

But, since the ellipse is tangent to the circle, the two values of x in (3) must be equal; therefore, $(a^2-b^2)\,(4-4a^2-z^2)=b^2z^2\;\ldots\ldots\ldots\ldots(4).$

From (4) we find $\quad a^2b^2=a^4-\dfrac{a^4z^2}{4\,(1-a^2)}=$ a maximum$\;\ldots\ldots\ldots\ldots(5).$

Differentiating (5), and reducing, we find

$$(2-a^2)\,z^2=8\,(1-a^2)^2\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6);$$

and from (6) we find $\quad z^2=\dfrac{8\,(1-a^2)^2}{2-a^2}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$

The value of z in (5) gives $\quad ab=\dfrac{a^3}{(2-a^2)^{\frac{1}{2}}}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$

Differentiating (7), we have $z\,dz = \dfrac{8a\,da}{(2-a^2)^2} - 8a\,da$ ....(9).

Making use of (8) and (9), and observing that, when $z = 0$, $a = 1$, and when $z = 2$, $a = 0$, we have, for the mean value of the elliptic disc,

$$\Delta = \int_0^2 \pi ab\,.\,2\pi z\,dz \div \int_0^2 2\pi z\,dz = \tfrac{1}{2}\pi \int_0^2 ab\,z\,dz$$

$$= 4\pi \int_0^1 \frac{a^4\,da}{(2-a^2)^{\frac{1}{2}}} - 4\pi \int_0^1 \frac{a^4\,da}{(2-a^2)^{\frac{5}{2}}}.$$

Let $a = \surd 2\ \sin\theta$; then $\Delta = 16\pi \int_0^{\frac{1}{2}\pi} \sin^4\theta\,d\theta - 4\pi \int_0^{\frac{1}{4}\pi} \tan^4\theta\,d\theta = \frac{1}{8}\pi\,(3\pi - 8).$

6798. (By Professor WOLSTENHOLME, M.A.)—If $\alpha, \beta, \gamma, \delta$ be the excentric angles of four points on an ellipse $(b^2x^2 + a^2y^2 = a^2b^2)$, prove that (1) the area of the triangle whose corners are the vertices of the quadrangle is

$$\pm 4ab \frac{\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\alpha-\delta)\sin\frac{1}{2}(\gamma-\alpha)\sin\frac{1}{2}(\beta-\delta)\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\gamma-\delta)}{\sin\frac{1}{2}(\beta+\gamma-\alpha-\delta)\sin\frac{1}{2}(\gamma+\alpha-\beta-\delta)\sin\frac{1}{2}(\alpha+\beta-\gamma-\delta)};$$

(2) this is also the area of the triangle formed by the diagonals of the quadrilateral formed by the tangents at the four points, the two triangles being identical; and (3) if the normals at the four points meet in one point (X, Y), the area of the triangle is

$$ab\,\frac{\{[a^2X^2 + b^2Y^2 - (a^2-b^2)^2]^3 + 27a^2b^2(a^2-b^2)^2X^2Y^2\}^{\frac{1}{2}}}{(a^2-b^2)(a^2X^2 - b^2Y^2)}.$$

Solution by D. EDWARDES; BELLE EASTON; *and others.*

Substituting the values of the coordinates of the vertices in the expression in Cartesian coordinates for the area, we have $\Delta \div ab$ equal to a fraction whose numerator is

$$\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\gamma-\delta)\sin^2\tfrac{1}{2}(\alpha+\beta-\gamma-\delta) + \ldots + \ldots,$$

and denominator as in question. This denominator

$$= \pm(\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2\delta),$$

and from the biquadratics in $\cos\theta$ or $\sin\theta$ we see that this

$$= \pm c^{-4}(a^2X^2 - b^2Y^2) \text{.................................(1).}$$

Also, since $\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\gamma-\delta) + \ldots + \ldots$ vanishes, writing

$$1 - \cos(\alpha+\beta-\gamma-\delta) \quad\text{for}\quad 2\sin^2\tfrac{1}{2}(\alpha+\beta-\gamma-\delta), \text{ \&c.},$$

then expanding the cosines, we get for the numerator of $4\Delta \div ab$

$$-8\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\delta)\sin\tfrac{1}{2}(\alpha-\delta)\sin\tfrac{1}{2}(\beta-\delta)\sin\tfrac{1}{2}(\gamma-\delta)$$
$$-8\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\beta-\delta)\times\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\gamma-\delta)\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\gamma-\delta)$$
$$+2\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\gamma-\delta)\cos(\alpha-\delta)\cos(\beta-\gamma)$$
$$+2\cos(\alpha-\beta)\cos(\gamma-\delta)\sin\tfrac{1}{2}(\beta-\alpha)\sin\tfrac{1}{2}(\gamma-\delta),$$

reducing easily to

$16\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)\sin\frac{1}{2}(\alpha-\delta)\sin\frac{1}{2}(\beta-\delta)\sin\frac{1}{2}(\gamma-\delta)$, and therefore (1) is proved. Also, the triangle being self-conjugate, (2) follows from the harmonic properties of the quadrilateral. (3) If θ be the excentric angle of the foot of one of the normals from X, Y, then from the biquadratic in $\tan\frac{1}{2}\theta$, we find the equation whose roots are $\frac{1}{2}(t_1t_2+t_3t_4)$ &c. &c., and it is in fact $bY^2z^3+(a^2X^2+b^2Y^2-c^4)z+2ac^2X=0$. Forming the equation whose roots are the squares of the differences of the roots of this last, we have

$$\tfrac{1}{64}(\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\beta+\tan\tfrac{1}{2}\gamma\tan\tfrac{1}{2}\delta-\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\gamma-\tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\delta)^2(\;)^2(\;)^2$$
$$=\frac{4(a^2X^2+b^2Y^2-c^4)^3}{b^6Y^6}+\frac{108a^2c^4X^2}{b^4Y^4},\text{ that is}$$

$$\tfrac{1}{64}\sin^2\tfrac{1}{2}(\alpha-\beta)\sin^2\tfrac{1}{2}(\beta-\gamma)\sin^2\tfrac{1}{2}(\gamma-\alpha)\sin^2\tfrac{1}{2}(\alpha-\delta)\sin^2\tfrac{1}{2}(\beta-\delta)\sin^2\tfrac{1}{2}(\gamma-\delta)$$
$$\div\cos^6\tfrac{1}{2}\alpha\cos^6\tfrac{1}{2}\beta\cos^6\tfrac{1}{2}\gamma\cos^6\tfrac{1}{2}\delta$$
$$=\frac{4}{b^6Y^6}\left[(a^2X^2+b^2Y^2-c^4)^3+27a^2b^2c^4X^2Y^2\right]\;\ldots\ldots\ldots\ldots\ldots(2).$$

From the biquadratic which we can form in $\cos\frac{1}{2}\theta$, we have

$$\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma\cos\tfrac{1}{2}\delta=\frac{bY}{4c^2}\;\ldots\ldots\ldots\ldots\ldots(3).$$

From (1), (2), (3), we have the result in the question.

[The expression $(a^2X^2+b^2Y^2-c^4)^3+27a^2b^2c^4X^2Y^2$ vanishes when (X, Y) is a point on the evolute, and is the product of the nine different values of $(aX)^{\frac{2}{3}}+(bY)^{\frac{2}{3}}-c^{\frac{4}{3}}$.]

6797. (By Professor Townsend, F.R.S.)—In the irrotational strain of an incompressible substance in a tridimensional space, if the equipotential surfaces of the strain be the same as of the attraction for the ordinary law of the inverse square of the distance of a corresponding distribution of matter in the space, show that the potential of the strain will be in general of the same form as that of the attraction throughout the entire extent of the substance.

Solution by the Proposer.

For, if u and v be the two potentials, either of the strain and the other of the attraction, then, since they are by hypothesis, both constant and both variable together throughout the entire extent of the substance, therefore $v=f(u)$, and therefore

$$\nabla_2(v)=\frac{d^2v}{du^2}\cdot\left[\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2\right]+\frac{dv}{du}\cdot\nabla_2(u);$$

hence, since by the fundamental property of both potentials $\nabla_2(u)=0$ and $\nabla_2(v)=0$ throughout the entire extent of the substance, therefore $\frac{d^2v}{du^2}=0$ throughout the entire extent of the substance; and since consequently by integration $v=cu+c'$, where c and c' are both constants, therefore, &c.

7104. (By Professor Burnside, M.A.)—Show that (1) the general differential equation of lines of the second order may be put under the form $\dfrac{d^3}{dx^3}\left\{\left(\dfrac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right\}=0$; (2) $\dfrac{1}{2}\dfrac{d^2}{dx^2}\left\{\left(\dfrac{d^2y}{dx^2}\sin\omega\right)^{-\frac{2}{3}}\right\}=-K^{-\frac{2}{3}}$, where K is that absolute invariant of the conic which depends upon the size only, the area being πK; (3) hence also the general differential equation of all parabolas may be written

$$\frac{d^2}{dx^2}\left\{\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right\}=0.$$

Solution by Prof. Wolstenholme, M.A.; J. W. Sharpe, M.A.; *and others.*

If the equation of a line of the second order be

$$ax^2+by^2+c+2fy+2gx+2hxy=0,\quad ax+hy+g+(hx+by+f)\frac{dy}{dx}=0,$$

$$-\frac{d^2y}{dx^2}=\frac{(hx+by+f)\left(a+h\dfrac{dy}{dx}\right)-(ax+hy+g)\left(h+b\dfrac{dy}{dx}\right)}{(hx+by+f)^2}$$

$$=\frac{a(hx+by+f)^2+b(ax+hy+g)^2-2h(ax+hy+g)(hx+by+f)}{(hx+by+f)^3}$$

$$=\frac{(ab-h^2)(ax^2+by^2+c+2fy+2gx+2hxy)-\Delta}{(hx+by+f)^3},$$

where Δ is the discriminant. Hence

$$\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}=\frac{(hx+by+f)^2}{\Delta^{\frac{2}{3}}},$$

$$\frac{d}{dx}\left\{\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right\}=2\Delta^{-\frac{2}{3}}(hx+by+f)\left(h+b\frac{dy}{dx}\right)$$

$$=2\Delta^{-\frac{2}{3}}\left\{h(hx+by+f)-b(ax+hy+g)\right\}=2\Delta^{-\frac{2}{3}}\left\{(h^2-ab)x+hf-bg\right\},$$

whence
$$\frac{d^3}{dx^3}\left\{\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right\}=0;$$

and
$$\frac{1}{2}\frac{d^2}{dx^2}\left\{\left(\frac{d^2y}{dx^2}\sin\omega\right)^{-\frac{2}{3}}\right\}=\Delta^{-\frac{2}{3}}(h^2-ab)(\sin\omega)^{-\frac{2}{3}}=-K^{-\frac{2}{3}}.$$

This seems to be the compactest form in which the general differential equation can be obtained. I have never seen it so arranged before, but cannot answer for its being a novelty. Surely there must be some compact way of expressing the other absolute invariant, depending upon the shape only, in terms of the differential coefficients, but I have not succeeded in discovering any at all worth recording. If ρ be the radius of curvature, s the arc, and a, b the semi-axes,

$$\frac{9+\left(\dfrac{d\rho}{ds}\right)^2-3\rho\dfrac{d^2\rho}{ds^2}}{\rho^{\frac{4}{3}}}=9(ab)^{-\frac{2}{3}}=9K^{-\frac{2}{3}},$$

and is therefore
$$=-\frac{9}{2}\frac{d^2}{dx^2}\left\{\left(\frac{d^2y}{dx^2}\sin\omega\right)^{-\frac{2}{3}}\right\};$$

and $$\tfrac{1}{9}\left\{9+\left(\frac{d\rho}{ds}\right)^2\right\}+\frac{\left(\rho\frac{d^2\rho}{ds^2}\right)^2}{9+\left(\frac{d\rho}{ds}\right)^2-3\rho\frac{d^2\rho}{ds^2}}=H^2,$$

where H is the absolute invariant $=\dfrac{2-e^2}{(1-e^2)^{\frac{1}{2}}}$, and e the excentricity.

7105. (By Professor MINCHIN, M.A.)—If (a, b, s) denote the elongations and shear with reference to any pair of rectangular axes at a point P in a natural solid subject to uniplanar strain, and if k_1, k_2 denote the radii of gyration, about the axes, of any small area, S, surrounding P in the plane of strain; show that the quantity $ak_2^2+bk_1^2-2sp^2$ has the same value for all sets of axes at P, where Sp^2 denotes the product of inertia of S with respect to the axes.

Solution by G. EASTWOOD, M.A.; D. EDWARDES; *and others.*

Let K_1^2, K_2^2, P^2 refer to the new axes. Then

$$K_1^2=k_1^2\cos^2\theta+k_2^2\sin^2\theta-p^2\sin 2\theta,\quad K_2^2=k_1^2\sin^2\theta+k_2^2\cos^2\theta+p^2\sin 2\theta,$$
$$P^2=(k_1^2-k_2^2)\sin\theta\cos\theta+p^2\cos 2\theta,$$

Also (MINCHIN's *Statics*) $a'=a\cos^2\theta+b\sin^2\theta+s\sin 2\theta$,
$$b'=a\sin^2\theta+b\cos^2\theta-s\sin 2\theta,\quad s'=(b-a)\sin\theta\cos\theta+s\cos 2\theta.$$

Hence, substituting and reducing, we have
$$a'K_1^2+b'K_2^2-2s'P^2=ak_1^2+bk_2^2-2sp^2.$$

7147. (By Professor HUDSON, M.A.)—An arc of a curve is in the shape of the portion of a catenary cut off by any horizontal base: find the law of density of the arc so that its centre of gravity may bisect the line drawn from the vertex to the middle point of the base.

Solution by DR. CURTIS; W. J. C. SHARP, M.A.; *and others.*

In any curve symmetrical about a vertical axis Y, the centre of gravity of the curve intercepted between by any *two* horizontal lines will bisect the line joining the middle points of the intercepts on these lines, if the density ρ be proportional to $\dfrac{dy}{ds}$, that is, the sine of ϕ, the angle which the tangent to the curve makes with the horizon; for, in this case, $\bar{y}$ is given by the equation:—

$$\bar{y}\int_{y_0}^{y_1}\frac{dy}{ds}\,ds=\int_{y_0}^{y_1}y\frac{dy}{ds}\,ds,\quad\text{or}\quad\bar{y}\int_{y_0}^{y_1}dy=\int_{y_0}^{y_1}y\,dy,$$

or $$\bar{y} = \tfrac{1}{2}\left\{\frac{y_1^2-y_0^2}{y_1-y_0}\right\} = \frac{y_1+y_0}{2}.$$

In the case of the catenary,

$$\rho = \text{K}\frac{dy}{ds} = \text{K}\sin\phi = \frac{\text{K}s}{(s^2+c^2)^{\frac{1}{2}}},$$

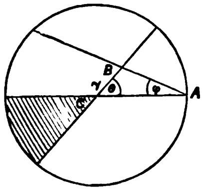
s denoting the length of the arc from the lowest point of the catenary and c the parameter of the curve.

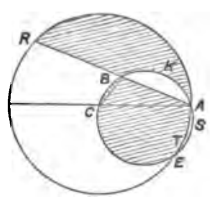
7135. (By D. Edwardes.)—Three points are taken at random upon the surface of a given circle. Prove that (1) the probability that the triangle formed by joining them contains the centre of the given circle, is $\frac{1}{4}$; and (2) the probability that the circle drawn through the random points, which may, of course, cut the given circle, does not contain the centre of the given circle, is $\frac{8}{3\pi^2}$.

Solution by the Proposer; Elizabeth Blackwood; *and others.*

(1) The probability (p) in each case is the same, if one point be fixed on the circumference of the given circle. Let A, Fig. I., be the fixed point, B a random point, C the centre of given circle, $\angle$ ACB $=\theta$, CB $=y$, CA $=a$. The shaded area will represent the number of favourable cases, and therefore

$$\pi^2a^4p = 2\int_0^{\pi}\int_0^{a}\tfrac{1}{2}a^2\theta\,.\,y\,d\theta\,dy, \text{ therefore } p = \tfrac{1}{4}.$$





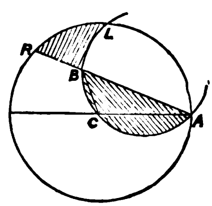


Fig. I. Fig. II. $y < a\cos\phi$. Fig. III. $y > a\cos\phi$.

(2) To avoid fractional expressions, refer B now to the point A. Let AB $=y$, CA $=a$, $\angle$ ACB $=\theta$, $\angle$ CAB $=\phi$, area BCEA $=u$, area RBKA $=v$, Fig. II.; and area BCA $=u'$, area RBL $=v'$, Fig. III.; and draw a circle through A, B, C. While $y < a\cos\phi$, the circle ABC cuts the fixed circle below the line CA, and above that line when $y > a\cos\phi$. If the second random point be situated in the shaded area, the circle through this point and B, A, does not contain C. Let R denote the radius of the circle ABC. Then R $= \frac{1}{2}a\,\text{cosec}\,(\theta+\phi)$,

$$u = \pi\text{R}^2 - \text{seg. BKA} - \text{seg. ATES} = \text{R}^2\left[\pi-\theta+\sin\theta\cos\theta-\pi\cos 2(\theta+\phi) + 2(\theta+\phi)\cos 2(\theta+\phi) - \sin 2(\theta+\phi)\right],$$

$$v = a^2\left(\tfrac{1}{2}\pi-\phi-\sin\phi\cos\phi\right) - \text{R}^2\left(\theta-\sin\theta\cos\theta\right);$$

whence, substituting functions of y and ϕ for functions of θ,

$$u+v = a^2(\pi-\theta-\phi-\sin\phi\cos\phi)+\tfrac{1}{2}\phi\operatorname{cosec}^2\phi\,(a^2\cos 2\phi-2ay\cos\phi+y^2)$$
$$+\tfrac{1}{2}\operatorname{cosec}\phi\,(2ay-a^2\cos\phi-y^2\cos\phi).$$

Also $$u' = \mathrm{R}^2(\pi-\theta+\sin\theta\cos\theta),$$

$$v' = \int_{\phi}^{\pi-\theta-\phi}[2a^2\cos^2\psi-2\mathrm{R}^2\sin^2(\theta+\phi-\psi)]\,d\psi = (a^2-\mathrm{R}^2)(\pi-\theta-2\phi)$$
$$-\tfrac{1}{2}a^2\sin 2(\theta+\phi)-\tfrac{1}{2}\mathrm{R}^2\sin 4(\theta+\phi)-\tfrac{1}{2}a^2\sin 2\phi+\tfrac{1}{2}\mathrm{R}^2\sin 2\theta;$$

hence

$$u'+v' = \tfrac{1}{2}\operatorname{cosec}\phi\,(2ay-y^2\cos\phi-a^2\cos\phi)+\tfrac{1}{2}\phi\operatorname{cosec}^2\phi\,(a^2-2ay\cos\phi+y^2)$$
$$+a^2(\pi-\theta-2\phi)-a^2\sin\phi\cos\phi.$$

The limits of ϕ are 0 and $\frac{1}{2}\pi$ and doubled. Then

$$\pi^2a^4p = 2\int_0^{\frac{1}{2}\pi}\int_0^{a\cos\phi}(u+v)\,y\,d\phi\,dy+2\int_0^{\frac{1}{2}\pi}\int_{a\cos\phi}^{2a\cos\phi}(u'+v')\,y\,d\phi\,dy, \quad\text{or}$$

$$\pi^2a^4p = \int_0^{\frac{1}{2}\pi}\int_0^{2a\cos\phi}[\operatorname{cosec}\phi(2ay-y^2\cos\phi-a^2\cos\phi)+2a^2(\pi-\theta-\phi-\sin\phi\cos\phi)$$
$$+\phi\operatorname{cosec}^2\phi\,(y^2-2ay\cos\phi)+a^2\phi\operatorname{cosec}^2\phi\cos 2\phi]\,y\,d\phi\,dy.$$

Now, $$\int_0^{2a\cos\phi}\theta y\,dy = \int_0^{2a\cos\phi}\tan^{-1}\frac{y\sin\phi}{a-\gamma\cos\phi}\,y\,dy$$
$$= a^2(\pi-2\phi)(1+\tfrac{1}{2}\cos 2\phi)-a^2\sin\phi\cos\phi,$$

therefore $$\pi^2p = \int_0^{\frac{1}{2}\pi}[\tfrac{4}{3}\sin 2\phi-\tfrac{4}{3}\cot\phi+4\cos^2\phi\,(\pi-\phi)+\tfrac{4}{3}\phi\operatorname{cosec}^2\phi$$
$$+2\phi+\tfrac{4}{3}\phi\cos 2\phi-2\pi-\pi\cos 2\phi]\,d\phi$$
$$= \tfrac{4}{3}+\int_0^{\frac{1}{2}\pi}[\tfrac{4}{3}\sin 2\phi+\pi\cos 2\phi-\tfrac{2}{3}\phi\cos 2\phi]\,d\phi$$
$$= \tfrac{8}{3},\quad\text{therefore}\quad p = \frac{8}{3\pi^2}.$$

7124. (By Professor GENESE, M.A.)—If three real triangles have the co-tangents of corresponding angles in arithmetical progression, prove that they must be similar.

Solution by E. W. SYMONS, M.A.; Prof. NASH, M.A.; *and others.*

Let (l, m, n), (λ, μ, ν), (l', m', n') be the three sets of cotangents, then

$$mn+nl+lm = 1,\quad \&c. = 1,\quad \&c. = 1\ \ldots\ldots\ldots\ldots(1, 2, 3),$$

and $$\lambda = \tfrac{1}{2}(l+l'),\quad \&c. = \&c.$$

Substituting in (2), we get, by means of (1) and (3),

$$mn'+m'n+\ldots\ldots = 2 = mn+m'n'+\ldots\ldots \text{ by (1) and (3),}$$

therefore $$(m-m')(n-n')+\ldots = 0.$$

Let $$l-l' = \kappa_1\ldots\ldots,$$

then we have $$\kappa_2\kappa_3+\ldots = 0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and, by (3), $$\kappa_1(m+n)+\ldots = 0,\ \text{ or }\ \kappa_1\sin^2\mathrm{A}+\ldots = 0\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Substituting for κ_3 its value from (4), we get

$$\kappa_1^2 \sin^2 A + \kappa_1 \kappa_2 (\sin^2 A + \sin^2 B - \sin^2 C) + \kappa_2^2 \sin^2 B = 0,$$

or, if a, b, c be sides of the triangle opposite A, B, C,

$\kappa_1^2 a^2 + 2\kappa_1 \kappa_2 ab \cos C + \kappa_2^2 b^2 = 0$, *i.e.*, $(\kappa_1 a + \kappa_2 b \cos C)^2 + (\kappa_2 b \sin C)^2 = 0$; therefore, for real values, $\kappa_1 = 0$, $\kappa_2 = 0$, and by symmetry $\kappa_3 = 0$, therefore $l = l'$, &c., &c., and the triangles are all equiangular and similar.

[The PROPOSER remarks that (l, m, n), &c. are the coordinates of points on the surface $yz + zx + xy = 1$, and the theorem follows immediately from the fact that this is *not a ruled surface*, that is, that three points on the surface cannot be in the same straight line.]

7148. (By Professor WOLSTENHOLME, M.A.)—Tangents are drawn to the curves $r^n \cos n\alpha = a^n \cos n(\theta + \alpha)$ from the point $r = a$, $\theta = 0$: prove that the points of contact lie on the curve

$$\left(\frac{r}{a}\right)^{n+1} \frac{\sin n}{\sin \theta} - \left(\frac{r}{a}\right)^n \frac{\sin (n+1)\theta}{\sin \theta} + 1 = 0,$$

which involves an extraneous factor $r^2 - ar\cos\theta - a^2$.

Solution by DR. CURTIS; PROF. MATZ, M.A.; *and others.*

In the class of curves defined by this equation, $-\dfrac{dr}{rd\theta} = \tan[n(\theta+\alpha)]$; consequently the angle between any radius vector and p the perpendicular from origin on the corresponding tangent $= n(\theta + \alpha)$; it is therefore geometrically evident that, in the case of tangents from the point referred to,

$$r\cos[n(\theta+\alpha)] = p = a\cos[(n+1)\theta + n\alpha],$$

therefore $\dfrac{r}{a} = \dfrac{\cos[(n+1)\theta + n\alpha]}{\cos[n(\theta+\alpha)]}$, and $\dfrac{r^n}{a^n} = \dfrac{\cos[n(\theta+\alpha)]}{\cos n\alpha}$;

hence
$$\frac{r^{n+1}}{a^{n+1}} = \frac{\cos[(n+1)\theta + n\alpha]}{\cos n\alpha},$$

therefore
$$\frac{r^n}{a^n}\sin[(n+1)\theta] - \frac{r^{n+1}}{a^{n+1}}\sin n\theta = \frac{\sin[(n+1)\theta]\cos[n(\theta+\alpha)] - \cos[(n+1)\theta + n\alpha]\sin n\theta}{\cos n\alpha},$$

or, by expanding the terms containing α, $= \sin[(n+1)\theta - n\theta] = \sin\theta$; hence the result required. It is geometrically evident that r', the distance of the extremity of the radius vector a, from any point on the curve, is given by the equation $r'^2 = a^2 + r^2 - 2ar\cos\theta$; and if, therefore, it is desired to exclude from the above locus the point from which the tangents are drawn, itself a point of contact of such a tangent, and corresponding to the condition $r' = 0$, it is plain that this factor must be suppressed.

[The PROPOSER calls this an extraneous factor, because all the other

points of contact lie on a single continuous curve not generally (if ever) passing through the point $(a, 0)$. Moreover, in his own investigation, he introduced this factor only to make the equation more compact. When $n = 2$, the locus is the straight line $2x + a = 0$; and, when $n = 3$, the hyperbola $3x^2 - y^2 + 2ax + a^2 = 0$.]

7107. (By Professor MALET, M.A.)—If the roots x_1, x_2, x_3, x_4, x_5, x_6 of the sextic equation $x^6 - p_1x^5 + p_2x^4 - p_3x^3 + p_4x^2 - p_5x + p_6 = 0$ be connected by the relation $x_1x_2x_3 = x_4x_5x_6$, prove that x_1, x_2, x_3 are the roots of the cubic

$$x^3 - \tfrac{1}{2}\left\{p_1 + \left(p_1^2 - 4p_2 + \frac{4p_5}{\sqrt{p_6}}\right)^{\frac{1}{2}}\right\}x^2$$

$$+ \tfrac{1}{2}\left\{\frac{p_5}{\sqrt{p_6}} + \left(\frac{p_5^2}{p_6} - 4p_4 + 4p_1\sqrt{p_6}\right)^{\frac{1}{2}}\right\}x - \sqrt{p_6} = 0.$$

Solution by G. G. MORRICE, B.A.; SARAH MARKS; *and others.*

Let $$x_1 + x_2 + x_3 = q, \qquad x_1x_2 + x_2x_3 + x_3x_1 = q_2,$$
$$x_4 + x_5 + x_6 = r, \qquad x_4x_5 + x_5x_6 + x_6x_4 = r_2,$$
$$x_1^2x_2^2x_3^2 = x_1x_2x_3x_4x_5x_6 = q_6.$$

We have $$p_1 = q_1 + r_1, \quad p_2 = q_1r_1 + q_2 + r_2, \quad p_3 = r_1q_2 + r_2q_1 + 2q_6,$$
$$p_4 = \sqrt{q_6}\,(r_1 + q_1) + r_2q_2, \quad p_5 = \sqrt{q_0}\,(r_2 + q_2).$$

Hence $$p_2 = \frac{p_5}{\sqrt{q_6}} + (p_1 - q_1)\,q.$$

Solving, $$q_1 = \tfrac{1}{2}\left\{p_1 \pm \left(p_1^2 - 4p_2 + \frac{4p_5}{\sqrt{p_6}}\right)^{\frac{1}{2}}\right\}.$$

Similarly $$q_2 = \tfrac{1}{2}\left\{\frac{p_5}{\sqrt{p_6}} \pm \left(\frac{p_5^2}{p_6} - 4p_4 + 4p_1\sqrt{p_6}\right)^{\frac{1}{2}}\right\},$$

which gives for finding x_1, x_2, x_3 the equation given in the Question.

7158. (By W. J. C. SHARP, M.A.)—Find (1) the law of density, varying as the distance from the centre, that the centre of gravity of a hemisphere may be at a distance $\frac{r}{n}$ from the centre; and (2) the surface density that the centre of gravity may be the same.

Solution by Dr. Curtis; Mary S. Meyer; *and others.*

1. Taking the centre of the sphere as origin and the axis of x perpendicular to the base of the hemisphere, and denoting by z the radius of any one of the homogeneous hemispherical shells of which the solid hemisphere is composed, and by ρ its density, we have $\rho = \mathrm{K}z^m$, and

$$\overline{\mathrm{X}}\int_0^r 2\pi z^2\,\mathrm{K}z^m\,dz = \int_0^r 2\pi z^2\,\mathrm{K}z^m\,\tfrac{1}{2}z\,dz,$$

or $$2\overline{\mathrm{X}}\int_0^r z^{m+2}\,dz = \int_0^r z^{m+3}\,dz,\ \text{therefore}\ \overline{\mathrm{X}} = \frac{(m+3)\,r}{2\,(m+4)};$$

but, by the condition of the question, we have

$$\overline{\mathrm{X}} = \frac{r}{n},\ \text{therefore}\ \frac{n}{2} = \frac{m+4}{m+3},\ \text{hence}\ m = \frac{8-3n}{n-2}.$$

2. Taking origin and axis of X as before, and axis of Y in the base of the hemisphere, and denoting by θ the angle subtended at the centre of the sphere by the radius of any one of the homogeneous rings into which the hemispherical shell may be divided by a number of consecutive planes parallel to its base, the density will be represented by $\mathrm{K}x^m$, or by $\mathrm{K}a^m\cos^m\theta$, and the volume of the ring by $2\pi yr\,d\theta\times\tau$, τ being the uniform thickness of the shell, or by $2\pi r^2\sin\theta\,d\theta\times\tau$, therefore

$$\overline{\mathrm{X}}\int_0^{\frac{1}{2}\pi} 2\pi\tau\,\mathrm{K}x^m\,r^2\sin\theta\,d\theta = \int_0^{\frac{1}{2}\pi} 2\pi\tau\,\mathrm{K}x^{m+1}\,r^2\sin\theta\,d\theta,$$

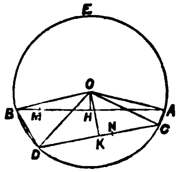
or $$\overline{\mathrm{X}}\int_0^{\frac{1}{2}\pi}\cos\ \ \theta\sin\theta\,d\theta = r\int_0^{\frac{1}{2}\pi}\cos^{m+1}\theta\sin\theta\,d\theta,\ \text{or}\ \overline{\mathrm{X}} = \frac{(m+1)\,r}{(m+2)};$$

but, by the condition of the question,

$$\overline{\mathrm{X}} = \frac{r}{n},\ \text{therefore}\ n = \frac{m+2}{m+1}\ \text{or}\ m = \frac{2-n}{n-1}.$$

6568. (By Professor Seitz, M.A.)—A quadrilateral is formed by joining the ends of two chords, each of which is drawn at random through a random point within a circle; show that the average area of the quadrilateral is $\dfrac{17r^2}{4\pi}$.

Solution by the Proposer.



Let M and N be the two random points, AB and CD the random chords through them, O the centre of the circle, and ABDC the quadrilateral. If the chords AB and CD intersect, the chords AC, AD, BC, BD will be the sides of the quadrilateral; and if C and D are both in the arc AEB, the chords AB, AD, CD, CB will be the sides. Draw OH, OK perpendicular to AB, CD.

Let $\mathrm{OA}=r$, $\angle\mathrm{AOH}=\theta$, $\angle\mathrm{COK}=\phi$, $\angle\mathrm{KOH}=\psi$, and ω = the angle OH makes with some fixed radius.

Then, from $\psi = 0$ to $\psi = \theta - \phi$, the area of the quadrilateral is

$$u_1 = \tfrac{1}{2}r^2 [\sin 2\phi - \sin 2\theta + 2\cos\psi \sin(\theta - \phi)];$$

from $\psi = \theta - \phi$ to $\psi = \theta + \phi$, it is $u_2 = 2r^2 \sin\theta \sin\phi \sin\psi$; and from $\psi = \theta + \phi$ to $\psi = \pi$, it is $u_3 = \tfrac{1}{2}r^2 [\sin 2\phi + \sin 2\theta - 2\cos\psi \sin(\theta + \phi)]$.

An element of the circle at M is $2r^2 \sin^2\theta \, d\theta$, at N it is $2r^2 \sin^2\phi \, d\phi$, and for elemental changes in the directions of AB and CD we have $d\omega$ and $d\psi$. The limits of θ are 0 and $\tfrac{1}{2}\pi$; of ϕ, 0 and θ, and doubled; of ω, 0 and 2π; and of ψ, as stated above, and doubled. Hence, since the whole number of ways the two chords can be drawn is $\pi^2 r^4 . \pi . \pi = \pi^4 r^4$, the average area of the quadrilateral is

$$\frac{4}{\pi^4 r^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\pi}\left\{\int_0^{\theta-\phi} u_1\,d\psi + \int_{\theta-\phi}^{\theta+\phi} u_2\,d\psi + \int_{\theta+\phi}^{\pi} u_3\,d\psi\right\} \times 2r^2\sin^2\theta\,d\theta \,.\, 2r^2 \sin^2\phi\,d\phi\,d\omega$$

$$= \frac{32r^2}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left\{(\pi - 2\theta)\sin\theta\cos\theta + (\pi - 2\phi)\sin\phi\cos\phi + 2\sin^2\theta + 2\sin^2\phi\right\} \times \sin^2\theta \sin^2\phi \, d\theta\, d\phi$$

$$= \frac{2r^2}{\pi^3}\int_0^{\frac{1}{2}\pi}\left\{8(\pi\theta - 2\theta^2)\sin\theta\cos\theta - 4(\pi - 2\theta)(2\sin^2\theta - 3\sin^4\theta) + 15\theta + 16\theta\sin^2\theta - 15\sin\theta\cos\theta - 26\sin^3\theta\cos\theta\right\}\sin^2\theta\,d\theta = \frac{17r^2}{4\pi}.$$

7130. (By J. J. Walker, M.A.)—Prove that the cotangent of the angle made by the right line through $(x_1 y_1 z_1)$, $(x_2 y_2 z_2)$ with the side BC of the triangle of reference ABC is equal to

$$[(z_1 - z_2)\cos C - (y_1 - y_2)\cos B] \div (x_1 - x_2)\sin A.$$

Solution by Rev. T. R. Terry, M.A.; Prof. Scheffer; *and others.*

If P and Q be the two points, and θ the angle between the lines PQ and BC, we have at once

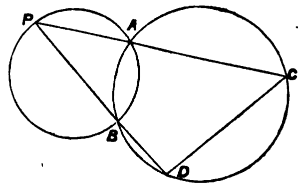
$$PQ \sin\theta = x_1 - x_2, \quad PQ\sin(B - \theta) = z_1 - z_2, \quad PQ \sin(C + \theta) = -(y_1 - y_2).$$

Therefore

$$(z_1 - z_2)\cos C - (y_1 - y_2)\cos B = PQ\{\cos C \sin(B - \theta) + \cos B \sin(C + \theta)\}$$
$$= PQ \sin A \cos\theta = (x_1 - x_2)\sin A \cot\theta.$$

7134. (By R. F. Scott, M.A.)—Two circles intersect at A and B. From any point P on one of them the lines PA, PB are drawn, and produced to meet the other circle again in C and D. Prove that, as the position of P varies, the straight line CD envelops a circle concentric with the circle ABDC.

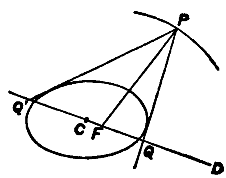
Solution by R. Knowles, B.A., L.C.P.; E. W. Symons, M.A.; *and others.*



The angle APB is evidently constant for all positions of P, therefore the difference of the arcs AB and CD in circle ABCD is constant, and therefore CD is constant, and will consequently touch at its mid-point a circle concentric with ABDC.

7136. (By T. Woodcock, B.A.)—Q is a point on the diameter parallel to the tangent at any point P of an ellipse. Prove that, if PQ always touches a given confocal ellipse, its length is constant.

Solution by E. W. Symons, M.A.; Christine Ladd, B.A.; *and others.*



The analytical proof is easy, but, assuming Quest. 6962, we may argue thus:—PF the normal at P bisects angle QPQ′ between two tangents from P to inner confocal, therefore

$$\frac{\text{FP}}{\text{QP}} = \frac{\mu}{\text{CD}},$$

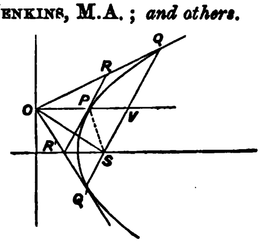
μ being a constant, and CD conjugate to CP,

therefore $\mu \, . \, \text{PQ} = \text{FP} \, . \, \text{CD} = \text{const.}$,

therefore $\text{PQ} = \text{const.}$

7153. (By R. F. Scott, M.A.)—A parabola always touches the rectangular axes so that its chord of contact is of constant length $2c$; prove that the locus of its focus is the curve $r = c \sin 2\theta$.

Solution by J. P. Johnston, B.A.; J. S. Jenkins, M.A.; *and others.*



The chord of contact QQ′ passes through the focus S, and is perpendicular to the line joining S to the origin O. Draw the diameter OPV through O,

$$\angle\theta = (\angle \text{SOQ}') = \angle \text{SQO} = \angle \text{QOV},$$

therefore $\angle \text{SVO} = 2\theta$,

$$\text{OS} = \text{OV} \sin 2\theta$$
$$= c \sin 2\theta.$$

6412. (By the Rev. T. P. Kirkman, M.A., F.R.S.)—The substitution 56781234 has twelve square roots, each having two circular factors of four elements. The powers and products of these twelve θ, ϕ, &c., viz., θ^2, $\theta\phi$, $\phi\theta$, &c., are a transitive group G of 32 substitutions, of which the title is $8 . 4 = 1 + 12_{44} + 13_{2222} + 6_{221111}$; $Q = 105$: *i.e.*, the group contains besides unity 12345678 and the 12, θ, ϕ, &c., 13 square roots of unity having no element undisturbed, and six square roots of unity having four undisturbed elements. The group of 32 has, including G, 105 equivalents, G, $\alpha G \alpha^{-1}$, $\beta G \beta^{-1}$, &c. Construct a transitive group H of the same order 32, having the same title, except that the number of its equivalents is $Q = 630$, in which the substitutions 12_{44} are not all square roots of the same substitution of the second order. [The interest of the question lies in the fact that this is the simplest of all cases of two transitive groups whose titles differ in nothing except the number Q.]

Solution by the Proposer.

The former of these groups is the following:—

12345678	12347856	14325876	14327658
A 23416785	C 23418567	43218765	43216587
α 34127856	34125678	32147658	32145876
B 41238567	D 41236785	21436587	21438765
56781234	G 78561234	58761432	I 76581432
E 67852341	85672341	87654321	J 65874321
78563412	H 56783412	76583214	K 58763214
F 85674123	67854123	65872143	L 87652143

The first half is an intransitive group of 16. The twelve substitutions marked with capitals are all square roots of α. The number of equivalents is 105, that of the substitutions of the form of α.

The second group is the following:—

12345678	12346587	12347856	12348765
α 21436587	21435678	ν 21438765	ρ 21437856
β 34127856	λ 34128765	34125678	ϕ 34126587
γ 43218765	μ 43217856	π 43216587	43215678
56781234	A 65871234	E 78561234	I 87651234
65872143	B 56782143	F 87652143	J 78562143
78563412	C 87653412	G 56783412	K 65873412
87654321	D 78564321	H 65874321	L 56784321

The first half is a group $1\alpha\beta\gamma$ followed by three derivates completing an intransitive group of 16. ABCD are four square roots of α, EFGH of β, and IJKL of γ.

The intransitive group of 16 can be written commencing with the group $1\alpha\lambda\mu$. There must therefore be another derivate completing a transitive group of 32, which has four square roots of each of α, λ, μ.

The same intransitive of 16 can be written commencing with the group $1\beta\pi\nu$, or with $1\nu\phi\rho$, or with $1\lambda\rho\pi$, or with $1\nu\mu\phi$; and with each of these arrangements is given a different derivate completing a transitive group of 32, containing four square roots of each substitution of the opening triplet $\beta\pi\nu$, &c.

The intransitive group of 16 has 105 equivalents. There are consequently $6 \cdot 105 = 630$ equivalent groups of 32.

The title of the former group is given in the abstract of my Treatise,

"The Complete Theory of Groups," at p. 144 of the *Proceedings of the Manchester Lit. Phil. Soc.*, 1863. It is no subject of wonder that the treatise was not printed, after the refusal of the Royal Society to print my much more original, if not more important theory of the Polyedra; but it is perhaps to be regretted that the work on Groups has disappeared from the archives of the Philosophical Society of Manchester, and cannot, after much enquiry, be found. I have no copy of my MS.

The title of the second group is given in a table of *Addenda*, at p. 172 of the same *Proceedings*, for 1865; but with the error Q = 210, instead of Q = 630. My oversight was the confinement of my attention to the two groups above noted $1\alpha\beta\gamma$ and $1\alpha\lambda\mu$.

All the six equivalent groups completed on the intransitive of 16 are obtained by the six multipliers 87654321, 87563421, 85672341, 85762431, 86754231, 86573241.

It was a surprise to me to find that two non-equivalent transitive groups may have titles identical in everything but the number Q.

6149. (By S. Tebay, B.A.)—Required a direct general solution of the equation $N^2 = H + 3st(2m-s-t)$, without assuming particular values of s, t to make the absolute term a square. [This problem solves the celebrated diophantine problem,—"To find n numbers such that, if each be taken from the cube of their sum, the remainders shall be cubes." Mr. Tebay's results are

$$24r^2(3mr^2+1)s = 9Hr^6-4, \quad (3r^2s+1)t = 2m-s.]$$

Solution by the Proposer; W. Nicholls, B.A.; *and others.*

Let $3st(2m-s-t) = 9r^2s^2t^2$; then $t = \dfrac{2m-s}{3r^2s+1}$. Substituting this value of t, we have

$$N^2(3r^2s+1)^2 = 9r^2s^4 - 36mr^2s^3 + 9r^2(Hr^2+4m^2)s^2 + 6Hr^2s + H = 9r^2(s^2-2ms+\tfrac{1}{3}Hr^2)^2.$$

Therefore $$24r^2(3mr^2+1)s = 9Hr^6-4.$$

This solves the problem. Its application to the diophantine problem in question is as follows.

Let a be the sum of n numbers, x, y, z, &c., such that a^3-x, a^3-y, a^3-z, &c. shall be cubes; then, by addition, $na^3-a = a^3(n-a^{-2})$ is the sum of n cubes, or $n-a^{-2} = n$ cubes.

Let $m-s$, $m-t$, $s+t-m$ be the roots of three of them; then, putting $H = n-(m^3+\text{remaining cubes})$, we have $a^{-2} = H+3st(2m-s-t)$, which is the proposed form.

This is a remarkable application of the diophantine analysis; but it is only so far general that $3st(2m-s-t)$ is always a square. The following particular solutions are deserving of notice. For s, t write $\lambda+s$, $\mu+t$, then

$$a^{-2} = H + 3\lambda\mu(2m-\lambda-\mu) + 3\mu(2m-2\lambda-\mu-s)s + 3(\lambda+s)(2m-\lambda-2\mu-s-t)t.$$

This equation can be solved whenever $H+3\lambda\mu(2m-\lambda-\mu)$ can be made a square.

Let $m=2,\ n=3,\ \lambda=1,\ \mu=1$; then

$$a^{-2}=1+3(1-s)s+3(1+s)(1-s-t)t.$$

Let $$P^2=1+3(1-s)s=(1+ps)^2;$$

then $$s=\frac{3-2p}{p^2+3},\quad P=\frac{3+3p-p^2}{p^2+3}.$$

Therefore $$a^{-2}=P^2+3(1+s)(1-s-t)t=(P+qt)^2;$$

therefore $$t=\frac{3(1-s^2)-2Pq}{q^2+3(1+s)},\quad a(P+qt)=1.$$

Let $p=q=1$; then $s=\frac{1}{4}$, $P=\frac{5}{4}$, $t=\frac{5}{76}$, $a=\frac{19}{25}=\cdot 76$, $x=\cdot 081065$, $y=\cdot 253783$, $z=\cdot 425152$.

Again, let $m=1,\ \lambda=\frac{1}{3},\ \mu=\frac{1}{3}$, then

$$a^{-2}=\tfrac{9}{4}+\tfrac{1}{12}(7-6s)s+\tfrac{1}{3}(1+3s)(4-3s-3t)t.$$

As before, we find

$$s=\frac{7-36p}{6(2p^2+1)},\quad P=\frac{9+7p-18p^2}{6(2p^2+1)},\quad t=\frac{\frac{1}{3}(1+3s)(4-3s)-2Pq}{q^2+1+3s},$$
$$a(P+qt)=1.$$

Let $p=\frac{1}{10}$, $q=0$, then $s=\frac{5}{9}$, $t=\frac{7}{9}$, $a=\frac{9}{14}$; and the three roots are $\frac{1}{28}$, $\frac{4}{28}$, $\frac{13}{28}$, which are Dr. HART's results.

6784. (By A. McINTOSH, B.A.)—Prove that (1) from any point of a Folium of Descartes which is not situated on the loop, two real tangents can be drawn to the curve; (2) if P be the point, PT, PT′ the tangents, and N the node, NT and NT′ make equal angles with the nodal tangents; (3) if PN be produced on to meet the chord TT′, it is bisected in N, and it also bisects TT′; (4) if the tangent (PQ) to the curve at the point P be produced on to meet the chord TT′ in Q, PQ is bisected by the curve; (5) the chord TT′ makes with the nodal tangents a triangle of constant area, and therefore touches a rectangular hyperbola having those tangents for asymptotes; (6) the rectangular hyperbola in (5) touches the curve at the vertex of the loop; (7) if the chord TT′ meet the loop in a second point R, PR being the line joining the primary point to this, PR also touches the rectangular hyperbola in (5); (8) the half of the chord TT′ remote from the loop subtends the same angle as the segment RT′ within the loop at the node; (9) the middle point of TT′ lies on an identical folium having its loop in the opposite quadrant; (10) if from the middle point of TT′ two tangents be drawn to the Folium in (9), the line joining the points of contact will pass through P; (11) the point of trisection of any tangent chord such as PT, most remote from the point of contact T, lies on a Folium of Descartes; (12) the centroid of the triangle PTT′ is such a point; (13) in nodal cubics in general, NT and NT′ form a harmonic pencil with the nodal tangents; (14) also TT′ and PR touch a conic section which touches the nodal tangents.

Solution by G. F. WALKER, M.A.; Prof. NASH, M.A.; *and others.*

The equation to the curve being $x^3+y^3=3axy$, let (α,β) be the point

P; then the tangent at P is $(\alpha^2-a\beta)\,x+(\beta^2-a\alpha)\,y=a\alpha\beta$, and the conic $(x^2-ay)\,\alpha+(y^2-ax)\,\beta=axy$ will pass through the points of contact of the tangents. The lines joining these points to N will be

$$3axy\,[\alpha x^2+\beta y^2-axy]=a\,(\alpha y+\beta x)\,(x^3+y^3),$$

and, remembering that $\alpha\beta$ is on the curve, this becomes, by eliminating a,

$$(\alpha y-\beta x)^2\,(\alpha x^2+\beta y^2)=0.$$

The equation to NT and NT′ is therefore $\alpha x^2+\beta y^2=0$, which proves (2), and they will be real if α and β have different signs, which proves (1).

The lines NT, NT′ will meet the curve at points lying on the conic $xy+\alpha y+\beta x=0$, and if $lx+my=1$ be the equation to TT′,

$$xy+(\alpha y+\beta x)\,(lx+my)=0$$

must be identical with $\alpha x^2+\beta y^2$; therefore

$$1+\beta m+\alpha l=0,\ \text{ and }\ \frac{\beta l}{\alpha}=\frac{\alpha m}{\beta}=\kappa \text{ suppose.}$$

Then $$\kappa\left(\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}\right)+1=0,\quad \text{therefore } \kappa=-\frac{1}{3a},$$

since $\alpha\beta$ is on the curve, and the equation to TT′ is $x\dfrac{\alpha}{\beta}+y\dfrac{\beta}{\alpha}=-3a$, and this touches $4xy=9a^2$, which proves (5).

This hyperbola passes through the point $x=y=\frac{3}{2}a$, the vertex of the loop, and has the same tangent $x+y=3a$, which proves (6).

The equation to PN is $\alpha y-\beta x$, and where this meets TT′ we have

$$x\left[\frac{\alpha}{\beta}+\frac{\beta^2}{\alpha^2}\right]=-3a,\ \text{ or }\ x=-\alpha,$$

which proves the first part of (3).

The conditions of the middle point of TT′ may be found thus: the x's of TT′ are found by eliminating y between

$$xy+\alpha y+\beta x=0\ \text{ and }\ \alpha x^2+\beta y^2=0,$$

and are therefore given by the equations

$$\alpha x^2+\frac{\beta^3x^2}{(x+\alpha)^2}=0,\quad \alpha\,(x+\alpha)^2+\beta^3=0,\quad x^2+2\alpha x+3a\beta=0.$$

The coordinate of the middle point is therefore $-\alpha$, which proves the rest of (3) and (9).

This also proves (10), as the two folia are symmetrical with respect to the lines $x\pm y$, and the chord will not only pass through P, but be bisected at it and be parallel to TT′.

The coordinates of Q are found to be $-\alpha\dfrac{\alpha^3-3\beta^3}{\alpha^3-\beta^3}$, $-\beta\dfrac{3\alpha^3-\beta^3}{\alpha^3-\beta^3}$, remembering that $\alpha\beta$ is on the curve, and therefore the coordinates of the middle point of PQ are $\dfrac{\alpha\beta^3}{\alpha^3-\beta^3}$, $-\dfrac{\alpha^3\beta}{\alpha^3-\beta^3}$, which is a point on the folium; which proves (4). The equation to the lines NT, NT′, NR is

$$x^3+y^3+xy\left[x\frac{\alpha}{\beta}+y\frac{\beta}{\alpha}\right]=0,\ \text{ or }\ (\alpha x^2+\beta y^2)\,(\beta x+\alpha y)=0.$$

The equation to NR is therefore $\beta x+\alpha y$, which proves (8), since the equation to the line joining N to the middle point of TT′ is $\beta x-\alpha y$, and these make equal angles with $x=0$, as also do NT and NT′.

The equation to RP is of the form

$$x\frac{\alpha}{\beta}+y\frac{\beta}{\alpha}+3a+\lambda(\beta x+\alpha y)=0,$$

and passes through (α, β).

Hence $\qquad \dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}+3a+2\lambda\alpha\beta=0$, therefore $\lambda=-\dfrac{3a}{\alpha\beta}$,

since (α, β) is on the curve; therefore PR is

$$x\frac{\alpha}{\beta}+y\frac{\beta}{\alpha}+3a-3a\left(\frac{x}{\alpha}+\frac{y}{\beta}\right)=0,\quad x\frac{\alpha^2-3a\beta}{\alpha\beta}+y\frac{\beta^2-3a\alpha}{\alpha\beta}+3a=0,$$

or, since (α, β) is on the curve, this becomes

$$x\frac{\beta^2}{\alpha^2}+y\frac{\alpha^2}{\beta^2}=3a,$$

which touches the hyperbola $4xy=9a^2$, which proves (7).

The coordinates of T are found, from $x(\alpha)^{\frac{1}{3}}=-y(-\beta)^{\frac{1}{3}}$ and the curve, to be $-\dfrac{\alpha^{\frac{2}{3}}+(-\beta)^{\frac{2}{3}}}{\alpha^{\frac{1}{3}}}$, $\dfrac{\alpha^{\frac{2}{3}}+(-\beta)^{\frac{2}{3}}}{(-\beta)^{\frac{1}{3}}}$, remembering that $\alpha\beta$ is on the curve; hence the coordinates of the required point of trisection are

$$\frac{\alpha^{\frac{2}{3}}-(-\beta)^{\frac{2}{3}}}{3\alpha^{\frac{1}{3}}},\quad \frac{\alpha^{\frac{2}{3}}-(-\beta)^{\frac{2}{3}}}{3(-\beta)^{\frac{1}{3}}},$$

which is a point on the folium $x^3+y^3+axy=0$, which proves (11).

The centroid of TPT′ evidently lies on this folium, since it is found by producing PN to $\frac{1}{3}$ of its length, [this proves (12)], and on the sinuous branch. The results (13) and (14) may be found by projection from (2), (5), and (7).

6858. (By Professor Wolstenholme, M.A.)—The hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ is cut at right angles by a concentric (not confocal) conic; prove that (1) the common chords which are not diameters touch the ellipse

$$\frac{x^2}{a^2}+\frac{y}{b^2}=\frac{1}{a^2-b^2};$$

(2) the locus of the intersection of normals to the hyperbola at the ends of such a common chord is the curve

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2\frac{x^2+y^2}{a^2+b^2}=\frac{a^2+b^2}{a^2-b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2,$$

whose asymptotes are $\qquad \dfrac{x}{a}\pm\dfrac{y}{b}\pm\left(\dfrac{a^2+b^2}{a^2-b^2}\right)^{\frac{1}{2}}=0.$

Solution by the Proposer; G. Heppel, M.A.; *and others.*

The equation of a conic cutting $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ at right angles will be

$$\frac{x^2}{a^2}+\frac{2xy}{ab}\lambda+\frac{y^2}{b^2}=\frac{a^2+b^2}{a^2-b^2};$$

or, putting $\lambda = \sin 2a$, we have, at their common points,

$$\frac{x^2}{a^2} + \frac{2xy}{ab}\sin 2a + \frac{y^2}{b^2} + \cos 2a\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \frac{a^2+b^2}{a^2-b^2} + \cos 2a,$$

which is $$\left(\frac{x}{a}\cos a + \frac{y}{b}\sin a\right)^2 = \frac{a^2\cos^2 a + b^2\sin^2 a}{a^2-b^2},$$

or is the equation of two common chords not diameters.

The envelope of these chords is

$$\left(\frac{x^2}{a^2} - \frac{a^2}{a^2-b^2}\right)\left(\frac{y^2}{b^2} - \frac{b^2}{a^2-b^2}\right) = \frac{x^2y^2}{a^2b^2}, \quad \text{or} \quad \frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2-b^2}.$$

The normals to the hyperbola are tangents to the orthogonal conic; hence their intersection is the pole of

$$\frac{x}{a}\cos a + \frac{y}{b}\sin a = \left(\frac{a^2\cos^2 a + b^2\sin^2 a}{a^2-b^2}\right)^{\frac{1}{2}}$$

with respect to the conic $\dfrac{x^2}{a^2} + \dfrac{2xy}{ab}\sin 2a + \dfrac{y^2}{b^2} = \dfrac{a^2+b^2}{a^2-b^2}$.

The equations for the point are, therefore,

$$\frac{\frac{x}{a} + \frac{y}{b}\sin 2a}{\cos a} = \frac{\frac{x}{a}\sin 2a + \frac{y}{b}}{\sin a} = \frac{a^2+b^2}{\left\{(a^2-b^2)(a^2\cos^2 a + b^2\sin^2 a)\right\}^{\frac{1}{2}}};$$

whence $$\frac{x}{a\cos a} = \frac{-y}{b\sin a} = \frac{a^2 \cdot b^2}{\cos 2a\left\{(a^2-b^2)(a^2\cos^2 a + b^2\sin^2 a)\right\}^{\frac{1}{2}}},$$

and the locus is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{1}{2}} = \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{\frac{x^2}{a^2} - \frac{y^2}{b^2}} \frac{a^2+b^2}{(a^2-b^2)^{\frac{1}{2}}}\left(\frac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{x^2+y^2}\right)^{\frac{1}{2}},$$

or $$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 \frac{x^2+y^2}{a^2+b^2} = \frac{a^2+b^2}{a^2-b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2;$$

and for the asymptotes we have

$$\left(\frac{x}{a} - \frac{y}{b}\right)^2 = \frac{(a^2+b^2)^2}{a^2-b^2}\left\{\frac{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2}{\left(\frac{x}{a} + \frac{y}{b}\right)^2(x^2+y^2)}\right\}_{\frac{x}{a}=\frac{y}{b}} = \frac{a^2+b^2}{a^2-b^2} \dots (1);$$

and, similarly, $$\left(\frac{x}{a} + \frac{y}{b}\right)^2 = \frac{a^2+b^2}{a^2-b^2} \dots\dots\dots\dots (2)$$

The curve consists of an acnode (quadruple point) at the origin and four hyperbolic branches.

7086. (By G. F. Walker, M.A.)—If the surface

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2Ax + 2By + 2Cz + D = 0$$

represent a paraboloid of revolution, prove that, provided f, g, h be finite,

its focus and directrix plane are given by the equations

$$f\left[x+\frac{2A}{a+b+c}\right]=g\left[y+\frac{2B}{a+b+c}\right]$$

$$=h\left[z+\frac{2C}{a+b+c}\right]=fgh\,\frac{2\frac{A^2+B^2+C^2}{a+b+c}-D}{2\,(Agh+Bhf+Cfg)},$$

$$2\,(Agh+Bhf+Cfg)(ghx+hfy+fgz)$$
$$+(A^2+B^2+C^2)\,fgh+D\,(g^2h^2+h^2f^2+f^2g^2)=0.$$

Solution by T. WOODCOCK, B.A.; D. EDWARDES; *and others.*

Putting $ax^2+\ldots+2fyz+\ldots+2Ax+\ldots+D$

$$\equiv(x-a)^2+(y-\beta)^2+(z-\gamma)^2-(lx+my+nz-p)^2,$$

where $l^2+m^2+n^2=1$, $(a\beta\gamma)$ being the focus, and $lx+my+nz=p$ the directrix plane, we get

$$\frac{a}{1-l^2}=\ldots=\ldots=\frac{-f}{mn}=\ldots=\ldots=\frac{A}{pl-a}=\ldots=\ldots=\frac{D}{a^2+\beta^2+\gamma^2-p^2}=\lambda\text{ say.}$$

Therefore $$a+b+c=2\lambda,$$

$$A^2+B^2+C^2=\lambda^2\left\{p^2+a^2+\beta^2+\gamma^2-2p\,(la+m\beta+n\gamma)\right\},$$

$$Agh+Bhf+Cfg=\lambda^3\,lmn\left\{p-(la+m\beta+n\gamma)\right\},$$

and $$g^2h^2+h^2f^2+f^2g^2=\lambda^4\,l^2m^2n^2;$$

therefore $$f\left(a+\frac{A}{\lambda}\right)=g\left(\beta+\frac{B}{\lambda}\right)=h\left(\gamma+\frac{C}{\lambda}\right)=-p\lambda\,lmn$$

$$=\frac{2\frac{A^2+B^2+C^2}{a+b+c}-D}{2\,(Agh+Bhf+Cfg)}\,fgh,$$

and the equation $lx+my+nz-p=0$ may be written in the required form by multiplying the left side by

$$2\lambda^5l^2m^2n^2\left\{p-(la+m\beta+n\gamma)\right\}.$$

5622. (By J. L. MCKENZIE, B.A.)—If s_r denote the sum of the r^{th} powers of the roots of the equation $x^n+p_1x^{n-1}+p_2x^{n-2}\ldots+p_n=0$; find the value of the determinant

$$\begin{vmatrix}
s_0 & s_1 & \ldots\ldots & s_{i-1} & s_{i+1} & \ldots\ldots & s_n \\
s_1 & s_2 & \ldots\ldots & s_i & s_{i+2} & \ldots\ldots & s_{n+1} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
s_{k-1} & s_k & \ldots\ldots & s_{k+i-2} & s_{k+i} & \ldots\ldots & s_{n+k} \\
s_{k+1} & s_{k+2} & \ldots\ldots & s_{k+i} & s_{k+i+2} & \ldots\ldots & s_{n+k+1} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
s_n & s_{n+1} & \ldots\ldots & s_{n+i-1} & s_{n+i+1} & \ldots\ldots & s_{2n}
\end{vmatrix}$$

Solution by R. F. Scott, M.A.; W. J. C. Sharp, M.A.; *and others.*

If there be n quantities $a_1, a_2 \dots a_n$, then $\Pi(a_p - a_q)$ is to mean the product of all the differences of the quantities $a_1 \dots a_n$. It is supposed that p is always less than q. This being so, we have

$$\begin{vmatrix} a_1^n & a_2^n & \dots\dots & a_n^n & x^n & 0 \\ a_1^{n-1} & a_2^{n-1} & \dots\dots & a_n^{n-1} & x^{n-1} & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ a_1 & a_2 & \dots\dots & a_n & x & 0 \\ 1 & 1 & \dots\dots & 1 & 1 & 0 \\ 0 & 0 & \dots\dots & 0 & 0 & 1 \end{vmatrix} = (-1)^n \Pi(a_p - a_q)(x-a_1)(x-a_2)\dots \dots(x-a_n),$$

$$\begin{vmatrix} a_1^n & a_2^n & \dots\dots & a_n^n & 0 & y^n \\ a_1^{n-1} & a_2^{n-1} & \dots\dots & a_n^{n-1} & 0 & y^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \\ a_1 & a_2 & \dots\dots & a_n & 0 & y \\ 1 & 1 & \dots\dots & 1 & 0 & 1 \\ 0 & 0 & \dots\dots & 0 & 1 & 0 \end{vmatrix} = (-1)^{n+1} \Pi(a_p - a_q)(y-a_1)(y-a_2)\dots \dots(y-a_n).$$

Multiply these two determinants by rows, $s_r = a_1^r + a_2^r + \dots + a_n^r$.

$$\begin{vmatrix} s_{2n} & s_{2n-1} & \dots\dots & s_n & x^n \\ s_{2n-1} & s_{2n-2} & \dots\dots & s_{n-1} & x^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ s_n & s_{n-1} & \dots\dots & s_0 & 1 \\ y^n & y^{n-1} & \dots\dots & 1 & 0 \end{vmatrix} = -\Pi(a_p - a_q)^2 . (x-a_1)(x-a_2)\dots(x-a_n) \times (y-a_1)(y-a_2)\dots(y-a_n).$$

Now let $f(z) = (z-a_1)(z-a_2)\dots(z-a_n) = z^n + p_1 z^{n-1} + \dots + p_n$.
Equate the coefficients of $x^i y^k$ on both sides, then, with the exception of an interchange of rows and columns,

$$\text{required determinant} = (-1)^{i+k} \Pi(a_p - a_q)^2 . p_{n-i}\, p_{n-k}.$$

7190. (By Professor Wolstenholme, M.A.)—If x, y, z be three quantities satisfying the two symmetrical equations

$$yz + zx + xy = 0, \quad x^3 + y^3 + z^3 + 4xyz = 0;$$

prove that (1) they will also satisfy one of the two pairs of semi-symmetrical expressions

$$y^2z + z^2x + x^2y = (y-z)(z-x)(x-y), = +xyz,$$
$$yz^2 + zx^2 + xy^2 = (y-z)(z-x)(x-y), = -xyz;$$

and (2) one set of the following equations will also be satisfied:—

$$(x^2 + yz - y^2 = 0, \quad y^2 + zx - z^2 = 0, \quad z^2 + xy - x^2 = 0);$$
$$(x^2 + yz - z^2 = 0, \quad z^2 + xz - x^2 = 0, \quad z^2 + xy - y^2 = 0).$$

Solution by Professor CAYLEY, F.R.S.

The two symmetrical equations represent a conic and a cubic respectively; they intersect therefore in 6 points, and if we denote by α a root of the equation $u^3+u^2-2u-1=0$, then the other two roots of this equation are $\beta, = -1-\frac{1}{\alpha}$, $\gamma = \frac{-1}{\alpha+1}$; viz., if $\alpha^3+\alpha^2-2\alpha-1=0$, then we have

$$(u-\alpha)\left(u+1+\frac{1}{\alpha}\right)\left(u+\frac{1}{\alpha+1}\right) = u^3+u^2-2u-1,$$

an identity which is easily verified. It may be remarked that, if $\phi\alpha = -1-\frac{1}{\alpha}$, then $\phi^2\alpha = \frac{-1}{\alpha+1}$, $\phi^3\alpha = \alpha$; the left-hand side of the last mentioned equation thus is $(u-\alpha)(u-\phi\alpha)(u-\phi^2\alpha)$, which remains unaltered when α is changed into $\phi\alpha$ or $\phi^2\alpha$. Then the coordinates of the six points of intersection can be expressed indifferently in terms of any one of the roots (α, β, γ), viz., the coordinates are

$$(\alpha^2-1,\ -\alpha,\ -1),\ (-1,\ \alpha^2-1,\ -\alpha),\ (-\alpha,\ -1,\ \alpha^2-1)\ldots(1,\ 2,\ 3),$$
$$(\alpha^2-1,\ -1,\ -\alpha),\ (-\alpha,\ \alpha^2-1,\ -1),\ (-1,\ -\alpha,\ \alpha^2-1)\ldots(4,\ 5,\ 6);$$

or they are equal to the like expressions in β and in γ respectively; say these are the coordinates of the points 1, 2, 3, 4, 5, 6 respectively, as shown by the attached numbers. Thus, writing $x, y, z = \alpha^2-1,\ -\alpha,\ -1$, we find $yz+zx+xy = \alpha-\alpha^2+1-\alpha^3+\alpha, = -(\alpha^3+\alpha^2-2\alpha-1) = 0$,

$$x^3+y^3+z^3+4xyz = (\alpha^2-1)^3-\alpha^3-1+4\alpha(\alpha^2-1),$$
$$= \alpha^6-3\alpha^4+3\alpha^3+3\alpha^2-4\alpha-2 = (\alpha^3+\alpha^2-2\alpha-1)(\alpha^3-\alpha^2+2), = 0,$$

which verifies the formulæ for the six points of intersection. Take, again, $x, y, z = \alpha^2-1,\ -\alpha,\ -1$; then we find

$$yz^2+zx^2+xy^2 = -\alpha-(\alpha^2-1)^2+\alpha^2(\alpha^2-1), = \alpha^2-\alpha-1,$$
$$y^2z+z^2x+x^2y = -\alpha^2+(\alpha^2-1)-\alpha(\alpha^2-1)^2, = -\alpha^5+2\alpha^3-\alpha-1.$$

Or, since $\alpha^3 = -\alpha^2+2\alpha+1$, and thence $\alpha^4 = 3\alpha^2-\alpha-1$, $\alpha^5 = -4\alpha^2+5\alpha+3$, the last equation becomes $y^2z+z^2x+x^2y = 2\alpha^2-2\alpha-2$. We have also

$$xyz = \alpha^3-\alpha, = -\alpha^2+\alpha+1;$$

hence the point in question is situate on each of the cubics

$$yz^2+zx^2+xy^2+xyz = 0,\quad y^2z+z^2x+x^2y+2xyz = 0,$$
$$y^2z+z^2x+x^2y-2(yz^2+zx^2+xy^2) = 0;$$

and this, of course, shows the points 1, 2, 3 are all three of them situate upon each of the three cubics; and in precisely the same manner it appears that the points 4, 5, 6 are all three of them situate on each of the three cubics $yz^2+zx^2+xy^2+2xyz = 0$, $y^2z+z^2x+x^2y+xyz = 0$,

$$yz^2+zx^2+xy^2-2(y^2z+z^2x+x^2y) = 0.$$

Again, from the values $x, y, z = \alpha^2-1,\ -\alpha,\ -1$, we have

$$x^2+yz-y^2 = 0,\quad y^2+zx-z^2 = 0,\quad z^2+xy-x^2 = 0;$$

viz., the point 1 lies on each of these conics; and similarly the point 2 lies on each of the same conics; and the point (3) lies on each of the same conics; that is, the conics in question have in common the points 1, 2, 3.

In like manner, the conics

$$x^2+yz-z^2 = 0,\quad y^2+zx-x^2 = 0,\quad z^2+xy-y^2 = 0,$$

have in common the points 4, 5, 6.

The general result is that the given conic and cubic meet in six points forming two groups of points (1, 2, 3) and (4, 5, 6); through the points (1, 2, 3) we have three cubics and three conics; and through the points (4, 5, 6) we have three cubics and three conics.

[If in the equation $x^3+x^2-2x-1=0$, whose roots are α, $\phi(\alpha)$, $\phi^2(\alpha)$, we put $x = 2\cos\theta$, the equation becomes

$$2(3\cos\theta+\cos 3\theta)+2(1+\cos 2\theta)-4\cos\theta-2=0,$$

or $$2\cos 3\theta+2\cos 2\theta+2\cos\theta=0,\quad\text{or}\quad \frac{\sin\frac{7}{2}\theta}{\sin\frac{1}{2}\theta}=0;$$

or the three roots are $2\cos\frac{2}{7}\pi$, $2\cos\frac{4}{7}\pi$, $2\cos\frac{6}{7}\pi$. The two equations $yz+zx+xy=0$, $x^3+y^3+z^3+3xyz=0$ are satisfied if $x:y:z=$ these three roots in any order, giving the six solutions. The semi-symmetrical systems are satisfied, the one by

$$x:y:z,\ \text{or}\ y:z:x,\ \text{or}\ z:x:y=\cos\tfrac{2}{7}\pi:\cos\tfrac{4}{7}\pi:\cos\tfrac{6}{7}\pi;$$

and the other by

$$z:y:x\ \text{or}\ y:x:z,\ \text{or}\ x:z:y=\cos\tfrac{2}{7}\pi:\cos\tfrac{4}{7}\pi:\cos\tfrac{6}{7}\pi.]$$

7169. (By Professor Hudson, M.A.)—The sum of the three sides of a right-angled spherical triangle is a quadrant, prove that the least value of the hypotenuse is $\cos^{-1}\frac{4}{5}$, and that in this case the spherical excess is $\sin^{-1}\frac{1}{9}$.

Solution by W. H. Blythe, M.A.; H. K. Moore, B.A.; *and others.*

Since the triangle is right-angled, and $a+b+c=\frac{1}{2}\pi$, we have

$$\cos c=\cos a\cos b=\cos a\sin(a+c)=\cos a\sin a\cos c+\cos^2 a\sin c;$$

hence $$\tan c=\sec^2 a-\tan a=(\tan a-\tfrac{1}{2})^2+\tfrac{3}{4};$$

therefore $\tan c$ is evidently a minimum when $\tan a=\frac{1}{2}$, which gives $\tan c=\frac{3}{4}$, or $c=\cos^{-1}\frac{4}{5}$.

In this particular case, also, we find

$$b=\tfrac{1}{2}\pi-\tan^{-1}\tfrac{1}{2}-\tan^{-1}\tfrac{3}{4}\ \text{or}\ b=\tan^{-1}\tfrac{1}{2}=a.$$

We find thus that $B=A$, and the sine of the spherical excess, or $\sin(A+B+C-\pi)$ reduces to $-\cos 2A$, C being a right angle; thus we obtain

$$1-2\cos^2 A=1-2\frac{\tan^2 b}{\tan^2 c}=1-\tfrac{2}{4}\cdot\tfrac{16}{9}=\tfrac{1}{9}.$$

7055. (By Professor Cavallin, M.A.)—From an origin, within a closed convex contour, a perpendicular p, making an angle θ with some fixed direction, is drawn on the tangent at P, show that, if ρ be the radius of

curvature at P, and ψ some function of θ, then is in general

$$\int_0^{2\pi}\left(p\psi-\frac{dp}{d\theta}\frac{d\psi}{d\theta}\right)d\theta=\int_0^{2\pi}\rho\psi\,d\theta.$$

[As special cases this formula gives the length of the curve when ψ = constant, its area when $\psi=p$, and for $\psi=v^{-1}$, where v is the velocity at P of a point moving in the orbit, the time of a revolution of the point. The more general formula, which may also be easily proved, is

$$\int_{\theta_1}^{\theta_2}\left(p\psi-\frac{dp}{d\theta}\frac{d\psi}{d\theta}\right)d\theta=\left(\psi\frac{dp}{d\theta}\right)_1-\left(\psi\frac{dp}{d\theta}\right)_2+\int_{\theta_1}^{\theta_2}\rho\psi\,d\theta.]$$

Solution by J. HAMMOND, M.A.; G. G. MORRICE, M.A.; *and others.*

Integrating by parts, we have

$$\int_{\theta_1}^{\theta_2}\frac{dp}{d\theta}\frac{d\psi}{d\theta}\,d\theta=\left(\psi\frac{dp}{d\theta}\right)_2-\left(\psi\frac{dp}{d\theta}\right)_1-\int_{\theta_1}^{\theta_2}\psi\frac{d^2p}{d\theta^2}\,d\theta.$$

From this and the well-known result $p+\dfrac{d^2p}{d\theta^2}=\rho$, the theorem in question follows immediately.

With limits 0 and 2π, the two values of $\psi\dfrac{dp}{d\theta}$ are identical, and

$$\int_0^{2\pi}\left(p\psi-\frac{dp}{d\theta}\frac{d\psi}{d\theta}\right)d\theta=\int_0^{2\pi}\rho\psi\,d\theta.$$

6952. (By Professor TOWNSEND, F.R.S.)—Two systems of forces $\Sigma(F_1)$ and $\Sigma(F_2)$ being supposed to act in a common space; show that the complete locus of the entire system of points in the space for which their principal moments have similar or opposite directions, consists of three right lines, one situated at infinity in the space, and all three lying in planes parallel to the two central axes of the systems.

Solution by the PROPOSER; W. J. C. SHARP, M.A.; *and others.*

The two groups of six constituents of the two systems of forces, to any arbitrary triad of rectangular coordinate planes in the space, being $X_1Y_1Z_1L_1M_1N_1$ and $X_2Y_2Z_2L_2M_2N_2$ for $\Sigma(F_1)$ and $\Sigma(F_2)$ respectively; we have manifestly, for every point xyz of the entire system constituting the complete locus in question, the two equations

$$\frac{L_1+Z_1y-Y_1z}{L_2+Z_2y-Y_2z}=\frac{M_1+X_1z-Z_1x}{M_2+X_2z-Z_2x}=\frac{N_1+Y_1x-X_1y}{N_2+Y_2x-X_2y}\quad\text{.........(1)},$$

which, as is evident *à priori* they ought, are easily seen to be satisfied by all points xyz at infinity for which

$$x:y:z=(mX_1+nX_2):(mY_1+nY_2):(mZ_1+nZ_2),$$

where m and n are any multiples, that is, by all points of the line of intersection with infinity of any plane parallel to the resultants R_1 and R_2, or, which is the same thing, to the central axes I_1 and I_2, of the two systems of forces; which line, as stated in the question, constitutes accordingly a part of the locus.

To determine the remainder of it. Denoting by k, a quantity to be subsequently determined, the common value of the three equivalents in the two equations (1), these two equations in xyz become replaced by the three in xyz and k,

$$\left.\begin{aligned}(L_1-kL_2)+(Z_1-kZ_2)\,y-(Y_1-kY_2)\,z&=0\\(M_1-kM_2)+(X_1-kX_2)\,z-(Z_1-kZ_2)\,x&=0\\(N_1-kN_2)+(Y_1-kY_2)\,x+(X_1-kX_2)\,y&=0\end{aligned}\right\}\;\ldots\ldots\ldots\ldots\ldots(2),$$

which, representing manifestly for every satisfying value of k a triad of planes passing through a common line in the space, show consequently that the remainder of the locus consists of as many such lines as there are satisfying values of k, the number and magnitudes of which are obtained immediately as follows.

Multiplying the three equations (2) by (X_1-kX_2), (Y_1-kY_2), (Z_1-kZ_2), respectively, and adding; we see at once that x, y, z go out together, and that k is given by the residual equation

$$(X_1-kX_2)(L_1-kL_2)+(Y_1-kY_2)(M_1-kM_2)+(Z_1-kZ_2)(N_1-kN_2)=0\,..(3),$$

which shows that there are but two satisfying values of k, and that the remainder of the locus consists in consequence of the two lines of intersection of the two concurrent triads of planes represented by the equations (2), when k_1 and k_2, the two roots of (3), are substituted in them successively for k.

That, as already shown for the third line at infinity, these two lines lie in two planes parallel to the resultants R_1 and R_2 of the two systems of forces, may be shown readily as follows. Multiplying the three equations (2) by X_2, Y_2, Z_2 respectively, and adding, we get at once, in xyz and k, the equation

$$\left.\begin{aligned}(Y_1Z_2-Z_1Y_2)\,x+(Z_1X_2-X_1Z_2)\,y+(X_1Y_2-Y_1X_2)\,z\\+(L_1-kL_2)\,X_2+(M_1-kM_2)\,Y_2+(N_1-kN_2)\,Z_2\end{aligned}\right\}=0\;\ldots\ldots(4),$$

which, for the two values of k given by (3), representing evidently two planes perpendicular to the direction $(Y_1Z_2-Z_1Y_2)$, $(Z_1X_2-X_1Z_2)$, $(X_1Y_2-Y_1X_2)$, and consequently parallel to the two directions $X_1Y_1Z_1$ and $X_2Y_2Z_2$; therefore, &c.

When the two systems of forces $\Sigma(F_1)$ and $\Sigma(F_2)$ in the space have each a single resultant; since then

$$L_1X_1+M_1Y_1+N_1Z_1=0 \quad\text{and}\quad L_2X_2+M_2Y_2+N_2Z_2=0,$$

the two roots of the quadratic (3) for k are respectively 0 and ∞, and the two aforesaid lines of the locus are given consequently by the two groups of equations

$$\left.\begin{aligned}&L_1+Z_1y-Y_1z=0,\;\; M_1+X_1z-Z_1x=0,\;\; N_1+Y_1x-X_1y=0\\\text{and}\;\;&L_2+Z_2y-Y_2z=0,\;\; M_2+X_2z-Z_2x=0,\;\; N_2+Y_2x-X_2y=0\end{aligned}\right\}\ldots(5),$$

which are those of the lines of action of the single resultants R_1 and R_2 of the two systems of forces $\Sigma(F_1)$ and $\Sigma(F_2)$ in the space; as it is manifest *à priori* they ought to be in that case.

7067. (By W. S. McCay, M.A.)—Show that the problem, "To construct a triangle of given species and minimum area with its vertices on

the sides of a given triangle (Δ)," admits of two solutions Δ_1 and Δ_2; and if the perpendiculars to the sides of Δ at the vertices of Δ_1 and Δ_2 meet in P_1 and P_2 respectively, then P_1, P_2 are inverse points to the circumscribed circle of Δ, and $\Delta_1 : \Delta_2 = OP_1 : OP_2$, where O is the centre of the circle, and Δ_1, Δ_2 are the two minima areas.

Solution by Professor WOLSTENHOLME, M.A.

I give the following construction for the two minimum triangles of given species to be inscribed in a given triangle (and at the same time for the two maximum triangles of the same species to be circumscribed to the same given triangle), but reserve the proof *in petto*, as it would involve considerable repetition of what I have given in the answer to the question of my own in the same number, and is mostly very elementary geometry which can readily be supplied.

Let ABC be the given triangle, and let A′, B′, C′ be the angles of the triangle to be inscribed; on the sides BC, CA, AB describe segments of circles containing angles $\pi - A'$, $\pi - B'$, $\pi - C'$ (on the parts towards the opposite angles in each case); these will meet in a point S, and straight lines drawn through A, B, C at right angles to SA, SB, SC will form a maximum triangle of the required species; and if a conic be inscribed in the triangle ABC having S for one focus, and s be its second focus, the feet of the perpendiculars from s on the sides of the triangle ABC will be the corners of a minimum inscribed triangle of the same required species, and having its sides parallel to those of the maximum circumscribed triangle. The other maximum and minimum are formed in the same way by describing segments of circles containing angles A′, B′, C′ which will meet in a point S′; and s' is found from S′ in just the same way as s from S. The areal coordinates of S, S′ are to be found from the equations

$$(S) \ldots X_1 (\cot A + \cot A') = Y_1 (\cot B + \cot B') = Z_1 (\cot C + \cot C'),$$

$$(S') \ldots X_2 (\cot A - \cot A') = Y_2 (\cot B - \cot B') = Z_2 (\cot C - \cot C').$$

Hence the areal coordinates of s, s' are to be found from

$$(s) \ldots \frac{x_1}{\frac{\sin A}{\sin A'} \sin (A + A')} = \frac{y_1}{\frac{\sin B}{\sin B'} \sin (B + B')} = \frac{z_1}{\frac{\sin C}{\sin C'} \sin (C + C')},$$

$$(s') \ldots \frac{x_2}{\frac{\sin A}{\sin A'} \sin (A - A')} = \frac{y_2}{\frac{\sin B}{\sin B'} \sin (B - B')} = \frac{z_2}{\frac{\sin C}{\sin C'} \sin (C - C')};$$

which are obviously in the same straight line with

$$\frac{x}{\sin 2A} = \frac{y}{\sin 2B} = \frac{z}{\sin 2C},$$

the centre of the circle ABC; and on substituting in the equation

$$a^2 (y_1 z_2 + y_2 z_1) + b^2 (z_1 x_2 + z_2 x_1) + c^2 (x_1 y_2 + x_2 y_1),$$

we, after some reduction, obtain an identity. Hence the two points s, s' are inverse points with respect to the circle ABC; and if O be the centre of the circle, Δ_1, Δ_2 the areas of the minimum triangles,

$$2\Delta_1 = (R^2 - Os^2) \sin A \sin B \sin C, \quad 2\Delta_2 = (Os'^2 - R^2) \sin A \sin B \sin C,$$

so that $$\Delta_1 : \Delta_2 = R^2 - Os^2 : Os'^2 - R^2,$$

or, since $R^2 = Os \,.\, Os'$, $\quad \Delta_1 : \Delta_2 = Os : Os'$.

There is an exceptional case when A′, B′, C′ are respectively equal to

A, B, C, s' being then at infinity; so that only one minimum triangle similar to ABC can be inscribed (with the angles on the corresponding sides). The point S′ may be any point on the circle ABC, and the corresponding maximum circumscribed triangles become both zero and indeterminate.

[The first result in question is an immediate consequence of the consideration that the inscription of triangles of given species (A′B′C′) in a given triangle (ABC) reduces to finding a point whose distances from the vertices of the latter are connected by the equations $ar_1 : br_2 : cr_3 = a' : b' : c'$, the two points which satisfy this being inverses to the circumscribed circle, as determined at the intersections of circles orthogonal thereto.]

5740. (By Professor WOLSTENHOLME, M.A.)—If $a > b > c > d$, prove that

$$\int_b^a \frac{dx}{\left[(a-x)(x-b)(x-c)(x-d)\left\{\frac{(a-x)(x-d)}{a-d}+\frac{(x-b)(x-c)}{b-c}\right\}\right]^{\frac{1}{2}}}$$

$$=\int_d^c \frac{dx}{\left[(a-x)(b-x)(c-x)(x-d)\left\{\frac{(a-x)(x-d)}{a-d}+\frac{(b-x)(c-x)}{b-c}\right\}\right]^{\frac{1}{2}}}.$$

[This question has been suggested by Quest. 5508.]

Solution by R. RAWSON; J. HAMMOND, M.A.; *and others.*

Consider the integral

$$u=\int_\beta^\alpha \frac{\{(x-a)(x-b)(x-c)(x-d)\}^{m-1}\,.\,dx}{\left\{\frac{(x-a)(x-d)}{a-d}-\frac{(x-b)(x-c)}{b-c}\right\}^{2m-1}} \quad\text{..............(1)},$$

which represents each side of the above equation when $m=\frac{1}{2}$.

In (1) put $x=\dfrac{(a-b)\,c+a\,(b-c)\,y}{a-b+(b-c)\,y}=a-\dfrac{(a-b)\,(a-c)}{a-b+(b-c)\,y}$ ....(2),

therefore

$$u=\frac{\{(b-c)(a-d)\}^{m-1}}{(-1)^{2m-1}}\int_{\frac{(a-b)(c-\beta)}{(b-c)(\beta-a)}}^{\frac{(a-b)(c-\alpha)}{(b-c)(\alpha-a)}} \frac{\left\{y\,(1-y)\left(y+\frac{(a-b)(c-d)}{(a-d)(b-c)}\right)\right\}^{m-1}.\,dy}{\left\{y^2+\frac{(a-b)(c-d)}{(a-d)(b-c)}\right\}^{2m-1}}$$

..............(3).

Let $u_1=u$, when $\alpha=\mathrm{A}$ and $\beta=\mathrm{B}$; then we have

$$u_1=\frac{\{(b-c)(a-d)\}^{m-1}}{(-1)^{2m-1}}\int_{\frac{(a-b)(c-\mathrm{B})}{(b-c)(\mathrm{B}-a)}}^{\frac{(a-b)(c-\mathrm{A})}{(b-c)(\mathrm{A}-a)}} \frac{\{y\,(1-y)(y+r)\}^{m-1}.\,dy}{\{y^2+r\}^{2m-1}} \text{......(4)},$$

where $\qquad r=\dfrac{(a-b)\,(c-d)}{(a-d)\,(b-c)}.$

Let $u_2 = u$, when $\alpha = A'$, and $\beta = B'$; then we have

$$u_2 = \frac{\{(b-c)(a-d)\}^{m-1}}{(-1)^{2m-1}} \int_{\frac{(a-b)(c-B')}{(b-c)(B'-a)}}^{\frac{(a-b)(c-A')}{(b-c)(A'-a)}} \frac{\{y(1-y)(y+r)\}^{m-1}\,.\,dy}{(y^2+r)^{2m-1}} \quad\ldots\ldots(5).$$

Transform (5) by the relation $yz = -r$. Therefore

$$u_2 = \frac{\{(b-c)(a-d)\}^{m-1}}{(-1)^{2m-1}} \int_{\frac{(c-d)(a-B')}{(a-d)(c-B')}}^{\frac{(c-d)(a-A')}{(a-d)(c-A')}} \frac{\{z(1-z)(z+r)\}^{m-1}\,.\,dz}{(z^2+r)^{2m-1}},$$

or
$$u_2 = \frac{\{(b-c)(a-d)\}^{m-1}}{(-1)^{2m-1}} \int_{\frac{(c-d)(a-B')}{(a-d)(c-B')}}^{\frac{(c-d)(a-A')}{(a-d)(c-A')}} \frac{\{y(1-y)(y+r)\}^{m-1}\,.\,dy}{(y^2+r)^{2m-1}} \quad\ldots\ldots(6).$$

Now, $u_1 = u_2$, providing the two following equations are satisfied, viz.,

$$\frac{(a-b)(c-A)}{(b-c)(A-a)} = \frac{(c-d)(a-A')}{(a-d)(c-A')},\quad \frac{(a-b)(c-B)}{(b-c)(B-a)} = \frac{(c-d)(a-B')}{(a-d)(c-B')} \quad\ldots(7).$$

Equation (7) is satisfied by $A=a$, $B=b$, $A'=c$, $B'=d$, and, if m is taken $\frac{3}{4}$, the truth of the equation in the question is manifest.

If $m = \frac{1}{2}$,

$$\int_B^A \frac{dx}{\{(x-a)(x-b)(x-c)(x-d)\}^{\frac{1}{2}}} = \int_{B'}^{A'} \frac{dx}{\{(x-a)(x-b)(x-c)(x-d)\}^{\frac{1}{2}}} \ldots(8),$$

where A, B, A′, B′ must satisfy (7).

[Equation (8) is an interesting relation in elliptic functions, but it is, probably, to be found amongst the many equations which have been investigated by writers on elliptical functions.]

6194. (By Professor MATZ, M.A.) — Three unequal homogeneous spheres being thrown into a given hemispherical bowl; determine their position when in equilibrium.

Solution by W. J. C. SHARP, M.A.; CHARLOTTE A. SCOTT; *and others.*

If r be the radius of the bowl, a, b, c those of the spheres, the position of equilibrium is that of a weightless triangle ABC, whose sides are $b+c$, $c+a$, and $a+b$, respectively; and which is loaded with weights proportional to a^3, b^3, c^3 at its angular points, and hung from the centre of the bowl by strings $r-a$, $r-b$, and $r-c$ in length. This condition is that G, the centre of gravity of the weights, lies vertically below O the centre of the bowl. If g_1 be the centre of gravity of the spheres at B and C,

$$Og_1^2 = r^2 - 2r\frac{b^4+c^4}{b^3+c^3} + \left(\frac{b^4-c^4}{b^3+c^3}\right)^2,\quad Ag_1^2 = a^2 + 2a\frac{b^4+c^4}{b^3+c^3} + \left(\frac{b^4-c^4}{b^3+c^3}\right)^2,$$

$$OG^2 = r^2 - 2r\frac{a^4+b^4+c^4}{a^3+b^3+c^3} + \frac{a^8+b^8+c^8-2a^4b^4-2b^4c^4-2c^4a^4}{(a^3+b^3+c^3)^2},$$

which determines the depth of the centre of gravity; and with OA $= r - a$ and AG $= \dfrac{b^3+c^3}{a^3+b^3+c^3}\,Ag_1$, the angle GOA and the inclinations of the other lines OB and OC to the vertical, may be similarly determined.

[The above values of Og_1, Ag_1 and OG may be obtained by the repeated application of the formula $AD^2 = AB^2\dfrac{CD}{BC} + AC^2\dfrac{BD}{BC} - BD\,.\,CD$, where ABC is a triangle, and D a point in BC.]

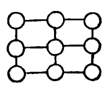
7039. (By T. C. Simmons, M.A.)—If two asymptotes of a cubic meet on the curve at a point of inflexion, prove that the third asymptote will pass through the same point.

7043. (By J. Hammond, M.A.)—A, B, C, D are four points on a cubic curve, BC meets the curve again in A′, CA in B′, AB in C′, AD in A″, BD in B″, CD in C″; prove that the lines A′A″, B′B″, C′C″ meet in a point which also lies on the curve.

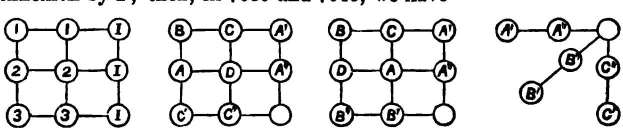
7058. (By J. J. Walker, M.A.)—Show that Mr. Hammond's Quest. 7043 may be generalized thus:—If conics pass through four given points on a cubic, the chords of their remaining points of intersection with it meet in one point, which also lies on the cubic.

Solution by J. Hammond, M.A.; T. Woodcock, M.A.; *and others.*

The following is a convenient notation for the graphic representation of theorems of collineation on a cubic curve:—Points of a cubic curve are represented by circles connected by straight lines; when three points are collinear, their connectors are in the same direction, thus O–O–O; when the names of the points are given, they may be inserted in the circles. The theorem that the connectors of collinear triads meet the curve again in a collinear triad, is represented thus



Let the three infinite points be denoted by 1, 2, 3, and the point of inflexion by I; then, in 7039 and 7043, we have



Questions 7043 and 7058 are examples of Prof. Sylvester's theory of residuation (*Reprint*, Vol. xxxiv., p. 34), the second residue being a single point, in whatever way it is arrived at. These theorems include a

number of others, by supposing two or more of the points to coincide; for example, if a conic touch a cubic in two points, the chord joining the remaining points of intersection meets the cubic at the third intersection of the satellite of the chord of contact.

[Of Question 7058, which is also solved in SALMON's *Higher Curves*, 2nd ed., p. 131, Art. 154, Mr. WALKER gives the following solution:—

If $\alpha\beta$, $\gamma\delta$ are two pairs of lines passing through the four points, then any conic passing through the same points is of the form $\alpha\beta = k\gamma\delta$, and the cubic must be of the form $\alpha\beta\epsilon = \gamma\delta\zeta$, so that $\epsilon = k\zeta$ is the chord of the remaining intersections of the conic with the cubic. But this, whatever k is, passes through the point $\epsilon = 0$, $\zeta = 0$, which is also on the cubic.]

6902. (By G. F. WALKER, M.A.)—A series of quadrics $\frac{x^2}{p}+\frac{y^2}{q}+\frac{z^2}{r}=1$ is drawn through a fixed point (α, β, γ): show that the locus of the centres of principal curvature at the fixed point is the surface

$$\alpha yz\,(x-\alpha)^2+\beta zx\,(y-\beta)^2+\gamma xy\,(z-\gamma)^2=0.$$

Solution by MARY S. MEYER; T. WOODCOCK, B.A.; *and others.*

If (ξ, η, ζ) be a centre of principal curvature at $(\alpha\beta\gamma)$, we have

$$(\xi-\alpha)\,\frac{p}{\alpha}=(\eta-\beta)\,\frac{q}{\beta}=(\zeta-\gamma)\,\frac{r}{\gamma}=\frac{\mathrm{R}}{\mathrm{K}},$$

where $\mathrm{K}^2=\frac{\alpha^2}{p^2}+\frac{\beta^2}{q^2}+\frac{\gamma^2}{r^2}$, and R is (SALMON, Art. 295) given by

$$\frac{\mathrm{K}^4}{\mathrm{R}^2}-\frac{\mathrm{K}}{\mathrm{R}}\left[\left(\frac{1}{q}+\frac{1}{r}\right)\frac{\alpha^2}{p^2}+\ldots+\ldots\right]+\left(\frac{1}{qr}\,\frac{\alpha^2}{p^2}+\ldots+\ldots\right)=0,$$

Hence we have $\left(1+\frac{\mathrm{R}}{\mathrm{K}}\right)\left(1+\frac{\mathrm{R}}{\mathrm{K}r}\right)\frac{\alpha^2}{p^2}+\ldots+\ldots=0,$

therefore $\frac{\eta\zeta}{\beta\gamma}\,(\xi-\alpha)^2+\ldots+\ldots=0$ is the required locus.

6974. (By A. McMURCHY, B.A.)—If $x+y+z=0$ and $a+b+c=0$, prove that $4\,(ax+by+cz)^3-3\,(ax+by+cz)\,(a^2+b^2+c^2)\,(x^2+y^2+z^2)$

$$-2\,(a-b)\,(b-c)\,(c-a)\,(x-y)\,(y-z)\,(z-x)=54abcxyz.$$

Solution by C. LEUDESDORF, M.A.; Prof. FRISBY, M.A.; *and others.*

$$4\,(\Sigma ax)^2-3\Sigma a^2\Sigma x^2=(\Sigma ax)^2-3\Sigma\,(bz-cy)^2=(\Sigma ax)^2-[\Sigma a\,(y-z)]^2,$$

$$=\tfrac{1}{3}\Sigma a\,(y-z)=4\,(bx+cy+az)(cx+ay+bz),$$

since $bz-cy=cx-az=ay-bx$; hence, if we write p for $(b-c)(c-a)(a-b)$,

and q for $(y-z)(z-x)(x-y)$, we have to prove that

$$2(ax+by+cz)(bx+cy+az)(cx+ay+bz) = pq+27abcxyz.$$

Now, since $p = bc(c-b)+\&c.$ and $-3abc = bc(c+b)+\&c.$,

therefore $\qquad 2\Sigma a^2b = -p-3abc, \quad 2\Sigma ab^2 = p-3abc,$

$$\therefore\ \Sigma a^2b\,\Sigma x^2y + \Sigma ab^2\,\Sigma xy^2 = \tfrac{1}{4}\left[(p+3abc)(q+3xyz)+(p-3abc)(q-3xyz)\right]$$
$$= \tfrac{1}{2}(pq+9abcxyz).$$

Multiply each side by 3, and then add to each $\Sigma a^3\,\Sigma x^3+18abcxyz$, or its equivalent $27abcxyz$; then

$$(ax+by+cz)^3+(bx+cy+az)^3+(cx+ay+bz)^3 = \tfrac{1}{2}(3pq+81abcxyz).$$

But the sum of the cubes on the left-hand side is equal to thrice the product of $ax+by+cz$, $bx+cy+az$, $cx+ay+bz$, since the sum of these quantities is zero; we have therefore what is required.

[Putting $a=0$, that is, substituting in sinister $a=0$, $b=-c$, $x=-(y+z)$, this sinister vanishes, therefore a is a factor thereof; and in the same way b, c, x, y, z are also factors; moreover, as the expression is homogeneous and of 6 dimensions, there is no other *literal* factor; and by putting the sinister $=\mathrm{N}\,(abcxyz)$, and taking $a=b=x=y=1$, $c=z=-2$, we find the *numerical* value to be $\mathrm{N}=54$.]

7137. (By R. Knowles, B.A., L.C.P.)—Prove that

$$S_1 \equiv \frac{1}{2}-\frac{1}{12}+\frac{1}{30}-\ldots+\frac{(-1)^{n-1}}{2n(2n-1)} \text{ ad inf.} = \tfrac{1}{4}\pi-\tfrac{1}{2}\log 2;$$

$$S_2 \equiv \int_0^1 dx \log(1-x)^{\frac{1}{x}} = -\tfrac{1}{6}\pi^2.$$

Solution by the Rev. T. R. Terry, M.A.; J. O'Regan; *and others.*

$$S_1 = \int_0^1 dx\int_0^x (1-y^2+y^4\ldots)\,dy = \int_0^1 \tan^{-1}x\,dx = \tfrac{1}{4}\pi-\tfrac{1}{2}\log 2.$$

$$S_2 = \int_0^1 = -\int_0^1 dx\,(1+\tfrac{1}{2}x+\tfrac{1}{3}x^2\ldots) = -\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right) = -\tfrac{1}{6}\pi^2.$$

7116. (By W. J. C. Sharp, M.A.)—If there be any number of centres of electric force (h_1, k_1, l_1), (h_2, k_2, l_2), &c., having charges ϵ_1, ϵ_2, &c., respectively; prove that (1) the line of force at the point (x, y, z) whose distances from the centres are r_1, r_2, &c., touches the vector

$$\Sigma\frac{\epsilon_1(x-h_1)}{r_1^3}\,.\,i+\Sigma\frac{\epsilon_1(y-k_1)}{r_1^3}\,.\,j+\Sigma\frac{\epsilon_1(z-l_1)}{r_1^3}\,.\,k$$

at that point; and (2) this vector represents the force at the point in magnitude and direction.

Solution by J. W. SHARPE, M.A.; BELLE EASTON; *and others.*

The force exerted by ϵ_1 at the point ρ, upon a unit of electricity at ρ, is given by the vector

$$\pi(\rho-\rho_1)\times\frac{\epsilon_1}{r_1^3} = (\rho-\rho_1)\times\frac{\epsilon_1}{r_1^3} = \frac{\epsilon_1(x-h_1)}{r_1^3}.i+\frac{\epsilon_1(y-k_1)}{r_1^3}.j+\frac{\epsilon_1(z-l_1)}{r_1^3}.k;$$

whence, by the consideration of the whole system, the result follows.

6840. (By W. M. MEE, M.A.)—If from any point on a fixed ordinate of a parabola, three normals be drawn to the curve, prove that the centroid of the triangle whose corners are the three points where the normals cut the curve, is a fixed point on the axis.

Solution by E. BUSK, B.A.; J. M. REEVES, M.A.; *and others.*

Let $y^2 = 4mx$ and $x = a$ be the equations to the parabola and fixed ordinate, then (SALMON'S *Conics*, p. 194, ex. 9)

$$4am = 8m^2+y_1^2+y_1y_2+y_2^2 = 8m^2+y_1^2+y_1y_3+y_3^2$$
$$= 8m^2+y_2^2+y_2y_3+y_3^2 \quad\text{.....................(1, 2, 3).}$$

From (1) and (2), $\quad y_2+y_3 = -y_1$ ....(4),

and for the centroid

$$x_G = \tfrac{1}{3}(x_1+x_2+x_3) = \frac{1}{12m}\left[2(y_2^2+y_3^2)+2y_2y_3\right] = \tfrac{2}{3}(a-2m), \text{ from (3)};$$

$$y_G = \tfrac{1}{3}(y_1+y_2+y_3) = 0, \text{ from (4); therefore, \&c.}$$

7114. (By J. HAMMOND, M.A.)—If A, B, C, D denote the first minors of the determinant

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}, \text{ its value is } Aa+Bb+Cc+Dd = pqrs,$$
$$\text{where } s = a+b+c+d,\quad p = d+a-b-c,$$
$$q = d+b-c-a,\quad r = d+c-b-b;$$

prove that $\quad s^3 = \dfrac{(D+A-B-C)(D+B-C-A)(D+C-A-B)}{(A+B+C+D)^2},$

with similar expressions for p^3, q^3, r^3.

Solution by G. G. MORRICE, B.A.; SARAH MARKS, B.A.; *and others.*

If P, Q, R, S denote the corresponding functions of the large letters; viz., $D+A-B-C$, &c., we have

$$Pp = (D+A-B-C)(d+a-b-c) = aA+bB+cC+dD = pqrs,$$

for $$bA + aB + dC + cD = 0, \quad cA + dB + aC + bD = 0,$$
and $$dA + cB + bC + aD = 0,$$
by a property of determinants. Similarly $Qq = Rr = Ss = pqrs$,

therefore $$\frac{PQR}{S^2} = s^3, \quad \frac{QRS}{P^2} = p^3, \quad \text{\&c.}$$

6644. (By W. J. C. Sharp, M.A.)—When the catalecticant of a binary $2n$-ic $(a, b, c \ldots k, l)(x, y)^{2n}$ vanishes, prove that the n quantities to the sum of whose $2n^{\text{th}}$ powers the quantic is reducible are the factors of the canonizant of $(a, b, c \ldots k)(x, y)^{2n-1}$.

Solution by the Proposer.

By differentiation, if
$$(a, b, c \ldots k, l \between x, y)^{2n} \equiv (\lambda_1 x + \mu_1 y)^{2n} + (\lambda_2 x + \mu_2 y)^{2n} \ldots + (\lambda_n x + \mu_n y)^{2n},$$
we have $$ax + by \equiv \lambda_1^{2n-1}(\lambda_1 x + \mu_1 y) + \lambda_2^{2n-1}(\lambda_2 x + \mu_2 y) + \text{\&c.},$$
$$bx + cy = \lambda_1^{2n-2}\mu_1(\lambda_1 x + \mu_1 y) + \lambda_2^{2n-2}\mu_2(\lambda_2 x + \mu_2 y) + \text{\&c.},$$
$$cx + dy = \lambda_1^{2n-3}\mu_1^2(\lambda_1 x + \mu_1 y) + \lambda_2^{2n-3}\mu_2^2(\lambda_2 x + \mu_2 y) + \text{\&c., \&c.};$$
therefore $$\begin{vmatrix} ax+by, & bx+cy, & cx+dy \ldots \\ bx+cy, & cx+dy, & dx+ey \ldots \\ cx+dy, & dx+ey, & ex+fy \ldots \end{vmatrix}$$
vanishes identically if $\lambda_1 x + \mu_1 y = 0$ or $\lambda_2 x + \mu_2 y = 0$, &c., which proves the proposition.

5843. (By T. R. Terry, M.A.)—If the coordinates of the points A, B, C, D, be represented by $(a_1 a_2 a_3)$, and so on, prove that the equation to the tangent plane at A to the sphere which passes through A, B, C, D is
$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ x-a_1 & y-a_2 & z-a_3 \end{vmatrix} = P[\mu_1(x-a_1) + \mu_2(y-a_2) + \mu_3(z-a_3)] + Q[\lambda_1(x-a_1) + \lambda_2(y-a_2) + \lambda_3(z-a_3)],$$
where $$P = (b_1-a_1)(c_1-b_1) + (b_2-a_2)(c_2-b_2) + (b_3-a_3)(c_3-b_3),$$
$$Q = (d_1-c_1)(a_1-d_1) + \text{\&c.}$$
$$\lambda_1 = (b_2-a_2)(c_3-b_3) - (b_3-a_3)(c_2-b_2), \quad \lambda_2 = \text{\&c.}, \quad \lambda_3 = \text{\&c.},$$
$$\mu_1 = (d_2-c_2)(a_3-d_3) - (d_3-c_3)(a_2-d_2), \quad \mu_2 = \text{\&c.}, \quad \mu_3 = \text{\&c.}$$

Solution by Sarah Marks; *the* Proposer; *and others.*

In Sir W. R. Hamilton's *Lectures on Quaternions*, Art. 329, it is proved that, considered as quaternions,
$$AB \,.\, BC \,.\, CD \,.\, DA \,.\, AR = AR \,.\, DA \,.\, CD \,.\, BC \,.\, AB,$$

where R is a point on the tangent plane at A. Hence

$$\mathrm{S}\,(\mathrm{AB}\,.\,\mathrm{BC}\,.\,\mathrm{CD}\,.\,\mathrm{DA}\,.\,\mathrm{AR}) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

But $\qquad \mathrm{AB}\,.\,\mathrm{BC} = -\mathrm{P} + \lambda_1 i + \lambda_2 j + \lambda_3 k,$

and $\qquad \mathrm{CD}\,.\,\mathrm{DA} = -\mathrm{Q} + \mu_1 i + \mu_2 j + \mu_3 k,$

therefore, substituting in (1),

$$\mathrm{S}\,(-\mathrm{P}+\lambda_1 i+\ldots)(-\mathrm{Q}+\mu_1 i+\ldots)\,[(x-a_1)\,i+\ldots] = 0,$$

therefore $$-\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ x-a_1 & y-a_2 & z-a_3 \end{vmatrix} + \mathrm{P}\,[\mu_1(x-a_1)+\ldots] + \mathrm{Q}\,[\lambda_1(x-a_1)+\ldots) = 0,$$

which is the required result.

6812. (By Professor MUKHOPÂDHYÂY, M.A.)—Find the moment of inertia of (1) a circle, (2) a regular hexagon, in each case about an axis making an angle α with the plane of the figure.

Solution by E. BUSK, B.A.; E. RUTTER; *and others.*

Since the moments of inertia about more than one axis in the plane of the figure are the same, the moments about all axes in the plane are equal. If we take the axis of z perpendicular to the plane and passing through the centre, and any two rectangular axes in the plane as axes of x and y, we have

$$\mathrm{A} = \Sigma\, m\,(y^2+z^2) = \Sigma\, my^2 = \Sigma\, mx^2 = \mathrm{B}, \quad \mathrm{C} = \Sigma\, m\,(x^2+y^2) = 2\mathrm{A};$$

therefore moment about an axis making an angle α with the plane

$$= \mathrm{A}\cos^2\alpha + \mathrm{C}\sin^2\alpha = \mathrm{A}\,(\cos^2\alpha + 2\sin^2\alpha) = \mathrm{A}\,(1+\sin^2\alpha).$$

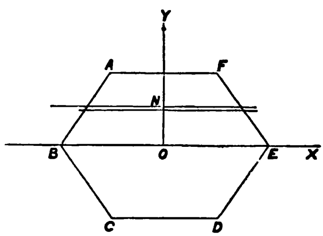
(1) In the case of the circle,

$$\mathrm{A} = \Sigma\, m\,(x^2) = \tfrac{1}{2}\Sigma\,(mr^2) = \tfrac{1}{2}\mu\int_0^a 2\pi r^3\,dr = \frac{\pi\mu a^4}{4} = \mathrm{M}\,.\,\tfrac{1}{4}a^2;$$

therefore required moment $= \mathrm{M}\,.\,\tfrac{1}{4}a^2(1+\sin^2\alpha)$.

(2) In the case of the hexagon,

$$\mathrm{AF} = a = \mathrm{OF},$$



$\mathrm{A} = \Sigma\,(my^2)$,

$$2\mu = \int_0^{a\frac{1}{2}\sqrt{3}} y^2\left(a + \frac{a\sqrt{3}-2y}{a\sqrt{3}}\,.\,a\right)dy$$

$$= 4\mu\int_0^{a\frac{1}{2}\sqrt{3}}\left(ay^2 - \frac{y^3}{\sqrt{3}}\right)dy$$

$$= 4\mu\left\{a\,.\,\frac{(a\frac{1}{2}\sqrt{3})^3}{3} - \frac{(a\frac{1}{2}\sqrt{3})^4}{4\sqrt{3}}\right\} = 4\mu a^4(\tfrac{1}{2}\sqrt{3})^3\left\{\tfrac{1}{3}-\tfrac{1}{8}\right\} = \frac{5\mu\sqrt{3}a^4}{16},$$

and $\mathrm{M} = 6\mu\,.\,a\,.\,\tfrac{1}{2}a\sqrt{3}$; therefore $\mathrm{A} = \frac{5}{24}\mathrm{M}\,.\,a^2$, therefore required moment $= \mathrm{M}\,\dfrac{5a^2}{24}(1+\sin^2\alpha)$.

6079. (By S. TEBAY, B.A.)—The late SAMUEL BILLS has stated (*Reprint*, Vol. XXII., p. 71) that "*any* number can be resolved into three rational cubes." Is this an established fact? Show that the equation $x^3+y^3+z^3=2\mu^3$ admits of a general solution; and thence find three numbers whose sum is unity, and such that, if each be taken from unity, the remainders shall be cubes.

Solution by the PROPOSER.

Let $x=aa_1$, $y=aa_2$, $z=aa_3$, $2\mu^3=n$; then $a^{-3}=n^{-1}(a_1^3+a_2^3+a_3^3)$. Take $a_3=p-s$, $a_1^3+a_2^3=nr^3+s^3$ (see *Reprint*, Vol. XXV., p. 31), then

$$a^{-3}=r^3+\frac{p^3}{n}-\frac{3p^2s}{n}+\frac{3ps^2}{n}=\left(r+\frac{ps^2}{nr^2}\right)^3, \text{ suppose};$$

then $$p=\frac{3nr^3s}{nr^3-s^3},\quad a=\frac{1}{r}\cdot\frac{nr^3-s^3}{nr^3+2s^3},\quad a_3=s\,\frac{2nr^3+s^3}{nr^3-s^3},$$

$$z=aa_3=\frac{s}{r}\cdot\frac{2nr^3+s^3}{nr^3+2s^3}.$$

We have now to solve $a_1^3+a_2^3=nr^3+s^3=2\mu^3r^3+s^3$.

Let $2\mu^3r^3+s^3=(\mu r+t)^3+(\mu r-t)^3=2\mu^3r^3+6\mu rt^2$. Therefore $s^3=6\mu rt^2$. Take $r=4\mu^2m^3$, and $t=3n^3$. Then $s=6\mu mn^2$. Therefore

$$a_1^3+a_2^3=(4\mu^3m^3+3n^3)^3+(4\mu^3m^3-3n^3)^3.$$

We can therefore take $a_1=4\mu^3m^3+3n^3$, and $a_2=4\mu^3m^3-3n^3$. Hence we have $$a=\frac{1}{8\mu^2m^3}\cdot\frac{16(\mu m)^6-27n^6}{8(\mu m)^6+27n^6},\quad x=\frac{4\mu^3m^3+3n^3}{8\mu^2m^3}\cdot\frac{16(\mu m)^6-27n^6}{8(\mu m)^6+27n^6},$$

$$y=\frac{4\mu^3m^3-3n^3}{8\mu^2m^3}\cdot\frac{16(\mu m)^6+27n^6}{8(\mu m)^6+27n^6},\quad z=\frac{3n^2}{4\mu m^2}\cdot\frac{32(\mu m)^6+27n^6}{8(\mu m)^6+27n^6}.$$

Let $\mu=1$; then $x^3+y^3+z^3=2$. This is the value of $3a^2-a$ when $a=1$. To satisfy the latter part of the problem, the ratio $\dfrac{s}{r}$ must be takon near its superior limit, which is $<\frac{3}{2}$. Take $m=7$, $n=4$; then we have

$$x=\frac{32667415}{45095239},\quad y=\frac{40691280}{45095239},\quad z=\frac{43298167}{45095239};$$

and the numbers are

$$1-\left(\frac{32667415}{45095239}\right)^3=\frac{34861358682288098698375}{91704802301923553136919},$$

$$1-\left(\frac{40691280}{45095239}\right)^3=\frac{67375818505225585152000}{91704802301923553136919},$$

$$1-\left(\frac{43298167}{45095239}\right)^3=\frac{8117242741633422423463}{91704802301923553136919}.$$

This subject is connected with the following general problem:—

"To find n numbers such that, if each be taken from the cube of their sum, the remainders shall be cubes" [which forms the first case of the general problem discussed by Dr. HART, under Quest. 4586, on pp. 66, 67 of Vol. xxvi. of our *Mathematical Reprints*].

Let $x_1, x_2, \ldots x_n$ be the numbers, a their sum; and suppose

$$a^3-x_1=a^3a_1^3,\quad a^3-x_2=a^3a_2^3,\quad \ldots,\quad a^3-x_n=a^3a_n^3;$$

then $$x_1=a^3(1-a_1^3),\quad x_2=a^3(1-a_2^3),\quad \ldots,\quad x_n=a^3(1-a_n^3);$$

and, adding, $x_1+x_2+\ldots+x_n=a=a^3(n-a_1^3+a_2^3-\ldots-a_n^3)$.

Therefore $$a^{-2} = n - a_1^3 - a_2^3 - \dots - a_n^3.$$

Let $n = 1$; then $a^{-2} = 1 - a_1^3$. Let $\frac{x}{y}$ be a value of a; then
$$a^3 - a = x(x^2 - y^2)\,y^{-3};$$
hence $x(x^2 - y^2) \equiv m^3$, which can be shown to be impossible. Thus, the continued product of three numbers in arithmetical progression cannot be a cube.

Let $n = 2$; then $a^{-2} = 2 - a_1^3 - a_2^3$. Let $a_2 = 1 - a_1$; then we have
$$a^{-2} = 1 + 3a_1 - 3a_1^2 = (1 - sa_1)^2, \text{ suppose};$$
therefore $a_1 = \dfrac{3 + 2s}{3 + s^2}$; hence, putting $s = -1$, we have
$$a_1 = \tfrac{1}{4}, \quad a_2 = \tfrac{3}{4}, \quad a = \tfrac{4}{5}, \quad x_1 = \tfrac{87}{125}, \quad x_2 = \tfrac{63}{125}.$$

Let $n = 3$; then $a^{-2} = 3 - a_1^3 - a_2^3 - a_3^3$. If $\frac{x}{y}$ be a value of a, $x(3x^2 - y^2)$ must be the sum of three cubes; x and y not being necessarily prime to one another. A general solution of this case is still a *desideratum*. From a sieve, or table, consisting of the sums of three cube numbers, we find, after the manner of Dr. Hart, the following triads of numbers:—

	x_1	x_2	x_3	Calculator.
$\frac{7}{12}$	$\frac{19}{96}$	$\frac{817}{4116}$	$\frac{6155}{32928}$	Tebay.
$\frac{7}{12}$	$\frac{97}{512}$	$\frac{813}{4096}$	$\frac{2401}{12288}$	Tebay.
$\frac{7}{12}$	$\frac{335}{1728}$	$\frac{4459}{23328}$	$\frac{9253}{46656}$	Tebay.
$\frac{8}{13}$	$\frac{511}{2197}$	$\frac{11080}{59319}$	$\frac{11627}{59319}$	Tebay.
$\frac{9}{14}$	$\frac{13}{49}$	$\frac{17}{64}$	$\frac{351}{3136}$	Hart.
$\frac{11}{18}$	$\frac{247}{1458}$	$\frac{8953}{39366}$	$\frac{8435}{39366}$	Tebay.
$\frac{19}{25}$	$\frac{1899}{15625}$	$\frac{4032}{15625}$	$\frac{4069}{15625}$	Lenhart.
$\frac{18}{29}$	$\frac{4501}{24389}$	$\frac{5805}{24389}$	$\frac{4832}{24389}$	Tebay.
$\frac{18}{31}$	$\frac{45927}{238328}$	$\frac{5768}{36041}$	$\frac{46313}{238328}$	Tebay.
$\frac{27}{44}$	$\frac{13851}{85184}$	$\frac{18954}{85184}$	$\frac{19467}{85184}$	Tebay.
$\frac{27}{46}$	$\frac{853}{4232}$	$\frac{17955}{97336}$	$\frac{19558}{97336}$	Tebay.

Dr. HART's numbers are still the smallest that have been found. It was the object of the present inquiry to ascertain if smaller ones existed, but a searching scrutiny has failed to detect them. Unless some larger value of a produce results which admit of very considerable reductions, we must be content with Dr. HART's simple fractions. It is remarkable that the value $a = \frac{7}{12}$ should produce three different triads of numbers; these are immediately obtained from $\frac{21}{36}$, $\frac{28}{48}$, $\frac{49}{84}$.

7196. (By Professor TOWNSEND, F.R.S.)—Perpendiculars being supposed let fall from the centre of an ellipsoid abc upon the several osculating planes to the curve of intersection of any two confocal quadrics $a_1b_1c_1$ and $a_2b_2c_2$; investigate symmetrically the equation of the cone they determine, and show from it that the several sections of the cone by parallels to the principal planes of the system are the evolutes of conics.

Solution by the PROPOSER; Dr. CURTIS; *and others.*

The equation of the osculating plane at any point (x, y, z) of the curve of intersection of $a_1b_1c_1$ and $a_2b_2c_2$ (see SALMON's *Geometry of Three Dimensions*, 4th ed., p. 329), being

$$a_1^{-2}a_2^{-2}a_3^2xx' + b_1^{-2}b_2^{-2}b_3^2yy' + c_1^{-2}c_2^{-2}c_3^2zz' = 1,$$

where $a_3b_3c_3$ is the third quadric of the system passing through (x, y, z), therefore, for every point (ξ, η, ζ) on the corresponding perpendicular,

$$a_1^2a_2^2\xi : b_1^2b_2^2\eta : c_1^2c_2^2\zeta = a_3^2x : b_3^2y : c_3^2z,$$

or, since $(a^2-b^2)(a^2-c^2)x = a_1^2a_2^2a_3^2$, &c., &c. (see same, p. 142),

$$a_1^4a_2^4\xi : b_1^4b_2^4\eta : c_1^4c_2^4\zeta$$
$$= (a^2-b^2)(a^2-c^2)x : (b^2-c^2)(b^2-a^2)y : (c^2-a^2)(c^2-b^2)z;$$

substituting from which for xyz in the familiar relation

$$a_1^{-2}a_2^{-2}x^2 + b_1^{-2}b_2^{-2}y^2 + c_1^{-2}c_2^{-2}z^2 = 0,$$

(see same, page 143), there results immediately between ξ, η, ζ the relation $\quad a_1^{\frac{2}{3}}a_2^{\frac{2}{3}}(b^2-c^2)^{\frac{2}{3}}\xi^{\frac{2}{3}} + b_1^{\frac{2}{3}}b_2^{\frac{2}{3}}(c^2-a^2)^{\frac{2}{3}}\eta^{\frac{2}{3}} + c_1^{\frac{2}{3}}c_2^{\frac{2}{3}}(a^2-b^2)^{\frac{2}{3}}\zeta^{\frac{2}{3}} = 0,$

which accordingly is the required equation, the form of which establishes at once the proposed property of the cone.

7180. (By A. MCMURCHY, B.A.)—Prove that the whole volume of the solid bounded by the surface

$$\left(\frac{x}{a}\right)^{\frac{2}{7}} + \left(\frac{y}{b}\right)^{\frac{2}{7}} + \left(\frac{z}{c}\right)^{2} = 1, \text{ is } \left(\frac{5.7}{9.11.13.17.19}\right).4\pi abc.$$

Solution by W. H. Blythe, M.A.; Sarah Marks; *and others.*

Consider the surface bounded by any surface of the form

$$\left(\frac{x}{a}\right)^{\frac{2}{n}}+\left(\frac{y}{b}\right)^{\frac{2}{n}}+\left(\frac{z}{c}\right)^{\frac{2}{n}}=1,$$

where n is an odd positive integer.

Take an elementary section at a distance z from the origin, perpendicular to axis z, take

$$1-\left(\frac{z}{c}\right)^{\frac{2}{n}}=\mu^2,\quad x=a\mu^n\cos^n\theta,\quad y=b\mu^n\sin^n\theta,$$

the area of the section is $4ab\mu^{2n}\,.\,\mathrm{E}$, where $\mathrm{E}=n\int_0^{\frac{1}{2}\pi}\cos^{n+1}\theta\sin^{n-1}\theta\,d\theta.$

Next, integrating $4ab\mu^{2n}\,.\,\mathrm{E}\,.\,dz$ twice from 0 to c, to find the volume of the whole solid, putting $z=c\sin^n\theta$, and therefore $\mu=\cos\theta$, we obtain, as final result, $4abc\mathrm{EB}$, where $\mathrm{B}=2n\int_0^{\frac{1}{2}\pi}\cos^{2n+1}\theta\sin^{n-1}\theta\,d\theta.$ Using the formulæ of reduction, we find that, when $n=7$,

$$\mathrm{E}=\frac{5\,.\,7\pi}{2^{11}},\quad \mathrm{B}=\frac{2^{12}}{9\,.\,11\,.\,13\,.\,17\,.\,19};$$

hence the volume required is as stated in the question.

[*Otherwise*:—$\mathrm{V}=8.\ \dfrac{abc}{(\frac{2}{7})^3}\,.\,\dfrac{[\Gamma(\frac{7}{2})]^3}{\Gamma(\frac{23}{2})}$ = result given, obtained at once by evaluating the Gamma Functions by means of

$$\Gamma(m)\,\Gamma(m+\tfrac{1}{2})=\frac{\pi^{\frac{1}{2}}}{2^{2m-1}}\Gamma(2m)].$$

6971. (By G. S. Carr, B.A.)—If a number of voltaic cells differing both in electro-motive force and conductivity be joined "in multiple arc," show that the difference of potentials of the electrodes of the battery, when connected by a wire, is equal to the sum of the current strengths of the separate cells divided by the sum of the conductivities of all parts of the circuit.

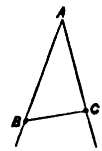
Solution by H. L. Orchard, M.A.; T. Woodcock, B.A.; *and others.*

If $e, e_1, \dots e_n$ be the different electro-motive forces, and $r, r_1, \dots r_n$ be the different resistances in the circuit, and C be the current strength, Ohm's Law gives $\mathrm{C}=\dfrac{\Sigma e}{\Sigma r}=\dfrac{\mathrm{E}}{\mathrm{R}}$ say; therefore $\mathrm{E}=\dfrac{\mathrm{C}}{\frac{1}{\mathrm{R}}}=\dfrac{\mathrm{C}}{\Sigma\frac{1}{r}}$,

since the cells are in *multiple* arc, and $\Sigma\dfrac{1}{r}$ is the sum of the conductivities

5490. (By S. TEBAY, B.A.)—AB, AC are two straight rods in a vertical plane, on which are two rings, B, C held in equilibrium by a flexible string passing through them, and having its ends fixed at A. If the equilibrium be slightly disturbed, find the time of a small oscillation.

Solution by the PROPOSER.



Let m, m' be the masses of B, C; T the tension of the string, l its length, α, α' the angles which AB, AC make with the vertical, BAC $= \Sigma$, ABC $= \phi$, ACB $= \phi'$, AB $= x$, AC $= y$, BC $= z$. Then, for the motions of B, C, we have

$$m\frac{d^2x}{dt^2} = gm\cos\alpha - 2\mathrm{T}\cos^2\tfrac{1}{2}\phi,$$

$$m'\frac{d^2y}{dt^2} = gm'\cos\alpha' - 2\mathrm{T}\cos^2\tfrac{1}{2}\phi';$$

eliminating T, $m\cos^2\tfrac{1}{2}\phi'\left(\frac{d^2x}{dt^2} - g\cos\alpha\right) - m\cos^2\tfrac{1}{2}\phi\left(\frac{d^2y}{dt^2} - g\cos\alpha'\right) = 0.$

By the geometry, $z\cos\phi = x - y\cos\Sigma$, $z\cos\phi' = y - x\cos\Sigma$;

therefore, eliminating ϕ, ϕ', we have, as our first equation (1),

$$m(-x\cos\Sigma + y + z)\left(\frac{d^2x}{dt^2} - g\cos\alpha\right) - m'(x - y\cos\Sigma + z)\left(\frac{d^2y}{dt^2} - g\cos\alpha'\right) = 0.$$

Let a, b, c be the values of x, y, z, when there is equilibrium, and assume $x = a + x'$, $y = b + y'$, $z = c + z'$; x', y', z' being small quantities, such that

$$x' + y' + z' = 0 \quad\text{.................................}(2).$$

Hence, neglecting small quantities of orders higher than the first, (1) becomes

$$mp\frac{d^2x'}{dt^2} - m'q\frac{d^2y'}{dt^2} = gmp\cos\alpha - gm'q\cos\alpha'$$
$$+ gm\cos\alpha(-x'\cos\Sigma + y' + z') - gm'\cos\alpha'(x' - y'\cos\Sigma + z');$$

putting $\quad l - 2a\cos^2\tfrac{1}{2}\Sigma = p, \quad l - 2b\cos^2\tfrac{1}{2}\Sigma = q.$

When the system passes through the position of equilibrium,

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = 0,$$

and (1) gives $gmp\cos\alpha - gm'q\cos\alpha' = 0$. Therefore

$$mp\frac{d^2x'}{dt^2} - m'q\frac{d^2y'}{dt^2} = gm\cos\alpha(-x'\cos\Sigma + y' + z')$$
$$- gm'\cos\alpha'(x' - y'\cos\Sigma + z') \text{ ......}(3).$$

Again, by the geometry, $z^2 = x^2 + y^2 - 2xy\cos\Sigma$,

or $\quad (c + z')^2 = (a + x')^2 + (b + y')^2 - 2(a + x')(b + y')\cos\Sigma;$

and observing that $c^2 = a^2 + b^2 - 2ab\cos\Sigma$, and neglecting small quantities of orders higher than the first,

$$cz' = (a - b\cos\Sigma)x' + (b - a\cos\Sigma)y' \text{ ....}(4).$$

From (2, 4), $x' = \dfrac{pz'}{2(a-b)\cos^2\frac{1}{2}\Sigma}$, $\quad y' = -\dfrac{qz'}{2(a-b)\cos^2\frac{1}{2}\Sigma}$.

Hence (3) becomes $\frac{d^2z}{dt^2} = 2g\cos^2\frac{1}{2}\Sigma\,\frac{pm\cos\alpha + qm'\cos\alpha'}{p^2m+q^2m'}$,

which shows that the system will oscillate in the same time as a simple pendulum whose length is

$$\frac{p^2m+q^2m'}{2\cos^4\frac{1}{2}\Sigma\,(pm\cos\alpha+qm'\cos\alpha')}.$$

4981. (By S. Tebay, B.A.)—A circular disc can revolve freely in a vertical plane about its axis, while a heavy particle descends in a curvilinear groove between two given points in its plane; find the brachystochrone, and assign the limits of possibility.

Solution by the Proposer.

Let the upper point be the origin, m the mass of the particle, m' the mass of the disc, and θ the angle through which it has turned in the time t. Then, by the principle of *vis viva*, we have

$$m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}+k^2m'\left(\frac{d\theta}{dt}\right)^2 = 2gmy;$$

or, putting $\qquad \frac{dy}{dx}=p,\quad \frac{d\theta}{dx}=p',\quad k^2\frac{m'}{m}=n^{-1},$

$$t=\int\left(\frac{1+p^2+n^{-1}p'^2}{2gy}\right)^{\frac{1}{2}}dx,\ \text{ and }\ \mathrm{V}=\left(\frac{1+p^2+n^{-1}p'^2}{y}\right)^{\frac{1}{2}}.$$

Since p, p' are independent, we have

$$\frac{d\,(\mathrm{V})}{dx}=\mathrm{N}p+\mathrm{P}\frac{dp}{dx}+\mathrm{P}'\frac{dp'}{dx},\quad \mathrm{N}-\frac{d\,(\mathrm{P})}{dx}=0,\quad \frac{d\,(\mathrm{P}')}{dx}=0.$$

Therefore $\qquad \mathrm{P}'=\text{const.}=c',\quad \frac{d\,(\mathrm{V})}{dx}=\frac{d\,(\mathrm{P}p)}{dx}+c'dp',$

and $$\mathrm{V}=\mathrm{P}p+c'p'+\frac{1}{c}.$$

Now, $\quad \mathrm{P}=\frac{p}{\{y(1+p^2+n^{-1}p'^2)\}^{\frac{1}{2}}},\quad \mathrm{P}'=\frac{n^{-1}p'}{\{y\,(1+p^2+n^{-1}p'^2)\}^{\frac{1}{2}}}=c'.$

Therefore $$\frac{1+n^{-1}p'^2}{\{y\,(1+p^2+n^{-1}p^2)\}^{\frac{1}{2}}}=c'p'+\frac{1}{c};$$

and, by division, $p'=ncc'$, and therefore $\theta=ncc'x$.

From these equations, we find

$$\frac{dy}{dx}=\left(\frac{c^2}{y}-nc^2c'^2-1\right)^{\frac{1}{2}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}.$$

Let a, b be the distances of the upper and lower points from the centre; α, β the angles which they make with the horizon; (x', y') the coordinates

of the point in the groove corresponding to (x, y). The distance of the point xy from the centre is

$$\left\{(x-a\cos\alpha)^2+(y-a\sin\alpha)^2\right\}^{\frac{1}{2}}=r,$$

which makes an angle $\sin^{-1}\left(\dfrac{y-a\sin\alpha}{r}\right)$ with the horizon. If the disc be turned back through the angle $\theta = ncc'x$, we have

$$x'=a\cos\alpha-r\cos\left\{\sin^{-1}\left(\frac{y-a\sin\alpha}{r}\right)-ncc'x\right\},$$

$$y'=a\sin\alpha+r\sin\left\{\sin^{-1}\left(\frac{y-a\sin\alpha}{r}\right)-ncc'x\right\}.$$

If we could eliminate xy between these equations and equation (A), we should obtain the equation to the brachystochrone required.

That the problem may be possible, it is necessary that

$$x=a\cos\alpha-b\cos(\beta+ncc'x),\quad y=a\sin\alpha+b\sin(\beta+ncc'x),$$

when the particle passes through the lower point. These equations combined with equation (A) furnish a relation between c, c'.

5710. (By E. W. Symons, M.A.)—If l_1, l_2, l_3, l_4 (of which l_4 is the greatest) be the semi-latus recta of the four conics circumscribing a triangle, each having one focus at the orthocentre, prove that

$$\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_3}>\frac{p_1+p_2+p_3}{\rho_2}>\frac{9}{l_4},$$

p_1, p_2, p_3 being the perpendiculars of the triangle, and ρ the radius of its self-conjugate circle.

Solution by J. A. Kealy, M.A.; *the* Proposer; *and others.*

If a, b, c be the sides of a $\triangle$, $a^2>(a-b+c)(a+b-c)$......,

therefore $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<\frac{1}{\left\{(a+b-c)(a-b+c)\right\}^{\frac{1}{2}}}+\ldots+\ldots;$$

but $$\frac{2}{\left\{(a+b-c)\cdot(a-b+c)\right\}^{\frac{1}{2}}}<\frac{1}{a+b-c}+\frac{1}{a-b+c},$$

... ...

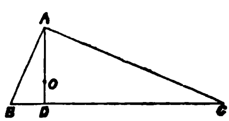
therefore $$\frac{1}{\left\{(a+b-c)(a-b+c)\right\}^{\frac{1}{2}}}+\ldots<\frac{1}{b+c-a}+\ldots,$$

hence, *à fortiori*, $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<\frac{1}{b+c-a}+\ldots+\ldots$$

Again, $(a+b)(b+c)(c+a)>8abc,$

or $(a+b+c)(bc+ca+ab)>9abc,$

or $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>\frac{9}{a+b+c};$$



$$\therefore \quad \frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c} > \frac{1}{a}+\frac{1}{b}+\frac{1}{c} > \frac{9}{a+b+c};$$

or, multiplying by 2Δ, we have, if r be radius of inscribed, and r_1, r_2, r_3 of escribed circumferences, $r_1+r_2+r_3 > p_1+p_2+p_3 > 9r$. Reciprocating this theorem with respect to the orthocentre, we shall get

$$\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_3} > \frac{1}{p'_1}+\frac{1}{\kappa_1}+\frac{1}{p'_2}+\frac{1}{\kappa_2}+\frac{1}{p'_3}+\frac{1}{\kappa_3} > \frac{9}{l^4},$$

($p'_1, p'_2, p'_3, \kappa_1, \kappa_2, \kappa_3$ being the distances of the orthocentre of the new Δ from its sides and vertices,)

or $\quad \dfrac{1}{l_1}+\ldots > \dfrac{p'_1+\kappa_1+\ldots}{p'_1 \cdot \kappa_1} > \dfrac{9}{l_4}, \quad$ or $\quad \dfrac{1}{l_1}+\dfrac{1}{l_2}+\dfrac{1}{l_3} > \dfrac{p_1+p_2+p_3}{\rho^2} > \dfrac{9}{l_4}.$

7176. (By R. F. Scott, M.A.)—Through the middle point of the radius vector of the lemniscate $r^2 = a^2 \cos 2\theta$, a perpendicular is drawn to the radius vector; prove that the locus of the point in which this meets the normal to the lemniscate, at the extremity of the radius vector, is the hyperbola $4r^2 \cos 2\theta = a^2$.

Solution by Dr. Curtis; J. O'Regan; *and others.*

More generally, let the curve be $r^n = a^n \cos(n\theta)$, all else being as stated in the question, then, as in this class of curves the radius vector is inclined to the normal at an angle $= n\theta$ (see solution of Quest. 7148), if ρ and ϕ be the polar coordinates of the locus required, $\phi = (1-n)\theta$, and $\rho \cos n\theta = \frac{1}{2}r$, therefore $2^n\rho^n \cos^n n\theta = r^n = a^n \cos n\theta$, or

$$(2\rho)^n = a^n (\cos n\theta)^{1-n} = a^n \cos\left(\frac{n}{1-n}\phi\right),$$

therefore $$\rho^{\frac{n}{1-n}} = (\tfrac{1}{2}a)^{\frac{n}{1-n}} \cos\left(\frac{n}{n-1}\phi\right)^{1-n}.$$

If $n = 2$, we have $\rho^{-2} = (\frac{1}{2}a)^{-2} \cos(2\phi)$, or $4\rho^2 \cos 2\phi = a^2$.

6182. (By D. Edwardes.)—The ends of a uniform rod of length $2a$ slide upon a smooth vertical circle (without inertia) of radius $\frac{2}{3}a\sqrt{3}$. If the system be set rotating about the fixed vertical diameter, prove that the inclination of the rod to the vertical at any time is given by the equation $\left(\dfrac{d\theta}{dt}\right)^2 = \dfrac{g\sqrt{3}}{a}(\sin\theta - \sin\alpha)$, α being the initial value of θ.

Solution by the PROPOSER; BELLE EASTON; *and others.*

The moment of inertia of rod in any position about the axis is $\frac{1}{3}a^2$. Let the lowest point of circle be origin, the fixed axis, the axis of z, and ϕ the angle described by the plane of the system in any time. Since the whole momentum about the axis is unchanged throughout the motion, we have $d\phi : dt =$ constant. Hence, if (x, y, z) be a point on rod distant r from its centre, we have

$$x = (\tfrac{1}{3}a\sqrt{3}\cos\theta + r\sin\theta)\cos\phi, \quad y = (\tfrac{1}{3}a\sqrt{3}\cos\theta + r\sin\theta)\sin\phi,$$
$$z = r\cos\theta + a - \tfrac{1}{3}a\sqrt{3}\sin\theta.$$

The equation of energy is

$$\tfrac{1}{2}\int dr\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} + g\int z\,dr = \text{constant},$$

or
$$a^2\left(\frac{d\theta}{dt}\right)^2 + 3ag\left(1 - \frac{1}{\sqrt{3}}\sin\theta\right) = C;$$

and initially $\theta = \alpha$, $\dfrac{d\theta}{dt} = 0$, therefore

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{g\sqrt{3}}{a}(\text{i n}\,\theta - \sin\alpha).$$

7109. (By J. J. WALKER, M.A.) — Show, from its fundamental property, that a principal axis (x) through any origin is determined by $V.\alpha\Sigma m\rho\,\alpha\rho = 0$, ρ being the vector of any element m of the body; and hence deduce some of the derived properties.

Solution by J. W. SHARPE, M.A.; Prof. MATZ, M.A.; *and others.*

The movement about any given vector λ is

$$\frac{\Sigma m\,T^2.V\lambda\rho}{T^2\lambda} = -\lambda^{-2}\Sigma m\left[\lambda^2\rho^2 - (S\lambda\rho)^2\right].$$

Now the function $\Sigma m\,(\lambda\rho^2 - \rho\,S\lambda\rho)$ is a self-conjugate linear vector function in λ; let it be denoted by $M\phi\lambda$, where M is a scalar constant; then

$$MS\lambda\,\phi\lambda = \Sigma m\left[\lambda^2\rho^2 - (S\lambda\rho)^2\right] = (\text{moment of inertia about }\lambda)\times T^2\lambda,$$
$$= L\times T^2\lambda, \text{ suppose;} \quad \text{therefore } S\lambda\,\phi\lambda = \frac{T^2\lambda}{M}L.$$

Hence, if λ satisfy the equation $S\lambda\,\phi\lambda = 1$, then $L = \dfrac{M}{T^2\lambda}$,

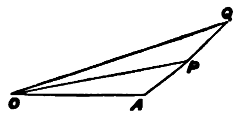
where the moment of inertia about the vector λ is denoted by L. Therefore the ellipsoid $S\lambda\,\phi\lambda = 1$ is the momental ellipsoid at the origin, and its axes are given by the equation $V\lambda\,\phi\lambda = 0$. This equation gives

$$V.\Sigma m\lambda\rho\,S\lambda\rho = 0 \text{ or } V.\Sigma m\,(\lambda^2\rho^2 - 2\lambda\rho\,S\lambda\rho) = 0,$$

$$\text{or} \quad V.\lambda\Sigma m\,(\lambda\rho^2 - 2\rho\,S\lambda\rho) = 0, \quad \text{that is,} \quad V\lambda\Sigma m\rho\,\lambda\rho = 0;$$

which is the equation required, having three vectors as resulting values for the variable λ, which give the directions of the three principal axes at the origin.

Let α denote a unit vector in the direction of one of the principal axes at O.



In the figure, let OA $= \alpha$, OP $= \rho$, and draw OQ in the plane of AOP, making

$$\angle \mathrm{QOP} = \angle \mathrm{POA},$$

and such that OA, OP, OQ have their lengths in continued proportion; then $-\rho\,\alpha\rho = \mathrm{OQ}$, and $\mathrm{T}\,.\,\mathrm{OQ} = \mathrm{T}^2\rho$; and hence the equation for α gives $\mathrm{V}\,.\,\alpha\Sigma m\,(\mathrm{OQ}) = 0$.

Now, let a particle of mass m be placed at Q; then Σm (OQ) denotes the vector of the centre of gravity of this system of particles; and the equation for α is the condition that this vector be collinear with α. Now, let the origin be at the centre of gravity, then $\Sigma m\rho = 0$. The equation to the principal axes at any other point σ is $\mathrm{V}\,.\,\lambda\Sigma m\,(\rho-\sigma)\,\lambda\,(\rho-\sigma) = 0$,

or $$\mathrm{V}\,.\,\lambda\Sigma m\rho\,\lambda\rho + \mathrm{V}\,.\,\lambda\,\Sigma m\sigma\,\lambda\sigma = 0.$$

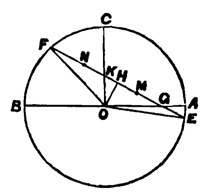
But $$\mathrm{V}\,.\,\lambda\Sigma m\sigma\,\lambda\sigma = \mathrm{M}\,.\mathrm{V}\lambda\sigma\,\lambda\sigma = 2\mathrm{MV}\lambda\sigma\,\mathrm{S}\lambda\sigma,$$

where M = mass of body; therefore $\mathrm{V}\,.\,\lambda\Sigma m\rho\,\lambda\rho + 2\mathrm{MV}\lambda\sigma\,\mathrm{S}\lambda\sigma = 0$ is the equation to the principal axes at σ.

But the equation reduces to $\mathrm{V}\,.\,\lambda\Sigma m\rho\,\lambda\rho = 0$, if σ be in one of the principal axes at the centroid, for then $\mathrm{V}\lambda\sigma = 0$, or if σ be in one of the principal planes, for then $\mathrm{S}\lambda\sigma = 0$; and these results show that their principal axes at the centroid are principal axes at all points of their length, and that at a point in one of the principal planes at the centroid one of the principal axes is the normal to the plane at that point.

5894. (By Professor SEITZ, M.A.)—A circle is divided into two semicircles, one of which is divided into two quadrants. If a point is taken at random in the surface of each quadrant, find the chance that the chord drawn through the points will be less than a line of given length.

Solution by the PROPOSER.



Let AOC and BOC be the two quadrants, M and N the random points, EF the chord through them, and O the centre of the circle. Draw OH perpendicular to EF.

Let 2β be the angle formed by the radii to the extremities of a chord equal to the given line. Let OA $= r$, KM $= x$, KN $= y$, KE $= u$, KF $= u_1$, KG $= v$, $\angle$ EOH $= \theta$, $\angle$ KOH $= \phi$.

Then $$u = r\,(\sin\theta + \cos\theta\tan\phi),$$

$$u_1 = r\,(\sin\theta - \cos\theta\tan\phi),\ \text{and}\ \ v = r\cos\theta\sec\phi\operatorname{cosec}\phi.$$

An element of the circle at M is $r\sin\theta\,d\theta\,dx$, and at N it is $(x+y)\,dy\,d\phi$.

1. When $\beta > \frac{1}{4}\pi$, while $\theta < \frac{1}{4}\pi$, the limits of ϕ are 0 and θ, and doubled; those of x are 0 and u; and those of y are 0 and u_1. From $\theta = \frac{1}{4}\pi$ to $\theta = \beta$, the limits of ϕ are 0 and $\frac{1}{2}\pi - \theta$, and doubled, while those of x and y are the same as above; and then they are $\frac{1}{2}\pi - \theta$ and θ, and doubled, while those of x are 0 and v, and those of y are 0 and u_1. Hence, since the whole number of ways the two points can be taken is $\frac{1}{16}\pi^2 r^4$, the required chance is

$$p = \frac{32}{\pi^2 r^4}\int_0^{\frac{1}{4}\pi}\int_0^{\theta}\int_0^{u}\int_0^{u_1} r\sin\theta\, d\theta\, d\phi\, dx\,(x+y)\,dy$$

$$+\frac{32}{\pi^2 r^4}\int_{\frac{1}{4}\pi}^{\beta}\left\{\int_0^{\frac{1}{2}\pi-\theta}\int_0^{u}\int_0^{u_1} d\phi\, dx\,(x+y)\,dy + \int_{\frac{1}{2}\pi-\theta}^{\theta}\int_0^{v}\int_0^{u_1} d\phi\, dx\,(x+y)\,dy\right\} \times r\sin\theta d\theta$$

$$= \frac{16}{\pi^2 r^3}\int^{\frac{1}{4}\pi}\int_0^{\theta}\int_0^{u}\left[(x+u_1)^2 - x^2\right]\sin\theta\, d\theta\, d\phi\, dx$$

$$+\frac{16}{\pi^2 r^3}\int_{\frac{1}{4}\pi}^{\beta}\left\{\int_0^{\frac{1}{2}\pi-\theta}\int_0^{u}\left[(x+u_1)^2 - x^2\right]d\phi dx + \int_{\frac{1}{2}\pi-\theta}^{\theta}\int_0^{v}\left[(x+u_1)^2 - x^2\right]d\phi dx\right\} \times \sin\theta\, d\theta$$

$$= \frac{32}{\pi^2}\int_0^{\frac{1}{4}\pi}\int_0^{\theta}(1-\cos^2\theta\sec^2\phi)\sin^2\theta\, d\theta\, d\phi$$

$$+\frac{16}{\pi^2}\int_{\frac{1}{4}\pi}^{\beta}\left\{\int_0^{\frac{1}{2}\pi-\theta} 2\,(1-\cos^2\theta\sec^2\phi)\sin\theta\, d\phi\right.$$

$$\left.+\int_{\frac{1}{2}\pi-\theta}^{\theta}\left[\sin\theta\cos^2\theta\,(\operatorname{cosec}^2\phi - \sec^2\phi) - \cos\theta\cos 2\theta\operatorname{cosec}\phi\sec\phi\right]d\phi\right\} \times \sin\theta\, d\theta$$

$$= \frac{32}{\pi^2}\int_0^{\frac{1}{4}\pi}(\theta - \sin\theta\cos\theta)\sin^2\theta\, d\theta + \frac{16}{\pi^2}\int_{\frac{1}{4}\pi}^{\beta}\left[(\pi-2\theta)\sin^2\theta - 2\sin\theta\cos^3\theta - \sin 2\theta\cos 2\theta\log\tan\theta\right]d\theta$$

$$= \frac{1}{\pi^2}\left\{8\beta(\pi-\beta) - \pi^2 - 4(\pi-2\beta)\sin 2\beta + 8\cos^4\beta - 4\sin^2 2\beta\log\tan\beta\right\}.$$

2. When $\beta < \frac{1}{4}\pi$, the required chance is found by changing the limits of θ in the first part of the above expression from 0 and $\frac{1}{4}\pi$ to 0 and β;

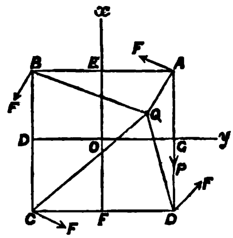
$$\text{therefore } p = \frac{32}{\pi^2}\int_0^{\beta}(\theta - \sin\theta\cos\theta)\sin^2\theta\, d\theta = \frac{2}{\pi^2}(2\beta - \sin 2\beta)^2.$$

When $\beta = \frac{1}{4}\pi$, both of the above results reduce to $p = \frac{1}{2} - \frac{2}{\pi} + \frac{2}{\pi^2}$.

7106. (By Professor Hudson, M.A.)—Four heavy equal rough particles at the corners of a square are connected by smooth rigid weightless wires and rest on a horizontal table. A force is applied to the system along the

wire joining two of the particles, and is increased until the particles are on the point of motion. Obtain an equation to determine the point about which the system begins to turn.

Solution by J. W. SHARPE, M.A.; CHARLOTTE A. SCOTT; *and others.*



Let ABCD be the square at the corners of which the particles are placed. Draw EOF, DOG through the middle points of the sides. Let AD be the line of action of the force; P the force, F the friction at each of the corners, Q the point about which the system begins to to turn. Then at each corner there is a force F acting at right angles to the line joining the corner to Q.

Now (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) denote the respective components of the force F at A, B, C, D; the x components being parallel to OE, and the y components parallel to OG.

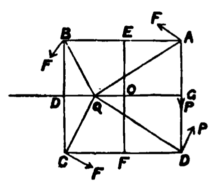
Then, by resolving and taking moments about Q,

$$y_1+y_2+y_3+y_4 = \mathrm{P}, \quad x_1+x_2+x_3+x_4 = 0, \quad r_1+r_2+r_3+r_4 = \frac{\mathrm{P}}{\mathrm{F}}p \ldots (1, 2, 3),$$

where r_1, r_2, r_3, r_4 denote the distances of A, B, C, D respectively from Q, and P the perpendicular from Q on AD.

To obtain a solution of these equations, we must avail ourselves of some considerations of symmetry, together with the fact that the solution must be unique.

Now, if we take into consideration two positions of Q corresponding to each other in the two quarters of the square AEOG and DFOG, it will be at once evident that equation (2) would be satisfied for one position if satisfied for the other; and that such would be the case also with equations (1) and (3), the values of the components of the frictions interchanging with one another in such a manner as not to change the value of $y_1+y_2+y_3+y_4$, and to leave $x_1+x_2+x_3+x_4$ zero for the 2nd position, if zero already for the 1st; also, such a change in the position of Q would have no effect upon $r_1+r_2+r_3+r_4$. But there exists no such symmetry with respect to the line EF. Hence the actual position of Q must be somewhere in the line DG.



Let it be at the point Q in OD, when $\mathrm{OQ} = r$.

Let $\quad 2a =$ length of side of square,

$\quad r_1 = \mathrm{QA}$ or QD, $\quad r_2 = \mathrm{QB}$ or QC;

then the equations of equilibrium are [from (1) and (3), (2) being identically satisfied]

$$\frac{2(a+r)}{r_1} - \frac{2(a-r)}{r_2} = \frac{\mathrm{P}}{\mathrm{F}}, \quad 2r_1 + 2r_2 = \frac{\mathrm{P}}{\mathrm{F}}(a+r).$$

Eliminating $\frac{\mathrm{P}}{\mathrm{F}}$, we have $\frac{(a+r)}{r_1} - \frac{a^2-r^2}{r_2} = r_1 + r_2$,

where $\quad r_1 = +[(a+r)^2+a^2]^{\frac{1}{2}}, \quad r_2 = +[(a-r)^2+a^2]^{\frac{1}{2}};$

and from this equation r must be found.

We shall now show that the equation has at least one real positive root. Writing it as follows—

$$(a+r)^2 r_2 - (a^2 - r^2)\, r_1 - r_1 r_2\, (r_1 + r_2) = 0,$$

and denoting the expression on the left by $f(r)$, we have, when $\frac{a}{r}$ is very small,

$$\frac{f(r)}{r^3} = \left(1+\frac{2a}{r}\right)\left(1-\frac{2a}{r}\right)^{\frac{1}{2}} + \left(1+\frac{a}{r}\right)^{\frac{1}{2}} - \left(1+\frac{2a}{r}\right)^{\frac{1}{2}} - \left(1-\frac{2a}{r}\right)^{\frac{1}{2}}$$

$$= \left(1+\frac{2a}{r}\right)\left(1-\frac{a}{r}\right) - \left(1-\frac{a}{r}\right) = \frac{2a}{r};$$

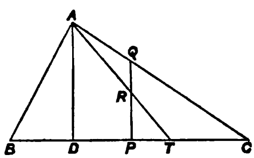
therefore $f(r) = 2ar^2$ when r is very large compared with a; therefore $f(\infty)$ and $f(-\infty)$ are both positive.

Also $$f(0) = a^2(\sqrt{2} - \sqrt{2} - 4\sqrt{2}) = -4\sqrt{2}a^3;$$

and therefore the equation has at least one positive real root.

6680. (By Professor MUKHOPÂDHYÂY, M.A.)—Trisect a given triangle by straight lines at right angles to the base.

Solution by Rev. T. C. SIMMONS, M.A.; E. RUTTER; *and others.*



Let T be one of the points of trisection of the base, and suppose it to lie between D and C. Take CP a mean proportional between CT and CD; then the perpendicular PQ will be one of the required lines. For, let it meet AT in R. Then since

$$CT : CP = CP : CD = CQ : CA,$$

therefore QT is parallel to AP; whence it easily follows that $\Delta ARQ = \Delta RPT$, and $\therefore\ \Delta CQP = \Delta CAT = \frac{1}{3}\Delta CAB$.

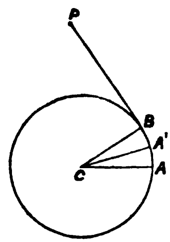
If t be the other part of trisection of BC, then, if it lie between C and D, p the foot of the other perpendicular may be found from the relation $cp^2 = ct \,.\, CD$. But if t lies between D and B we must use the relation $Bp^2 = Bt \,.\, BD$.

Thus the triangle is completely trisected.

4622. (By S. TEBAY.)—A fixed pulley rests on a smooth horizontal plane, and a string is fastened to its rim, and a weight (P) to the other end of the string, which lies on the plane, the string being tangential to the pulley. If the weight receives a given impulse perpendicular to the

string, determine the motion, and show that there is an epoch at which the string ceases to coil round the pulley. Assign the subsequent motion of P.

Solution by the PROPOSER.



Let CA be the initial radius, which in the time t has revolved into the position CA′, while the string BP is tangential to the pulley at B. Let x, y be the coordinates of P, CA $= a$, $\angle$ ACA′ $= \theta$, ACB $= \phi$, l the length of the string, BP $= r$, m the mass of the pulley, m' the mass of P, and v its initial velocity. Then, by the *conservation of areas*,

$$k^2 m\left(\frac{d\theta}{dt}\right)+m'\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=lm'v;$$

and, by the *conservation of vis viva*,

$$k^2 m\left(\frac{d\theta}{dt}\right)^2+m'\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}=m'v^2.$$

Now $\quad x=a\cos\phi-r\sin\phi,\qquad y=a\sin\phi+r\cos\phi,\qquad l=r+a(\phi-\theta),$

$$\frac{dx}{dt}=-r\cos\phi\frac{d\phi}{dt}-a\sin\phi\frac{d\theta}{dt},\quad \frac{dy}{dt}=-r\sin\phi\frac{d\phi}{dt}+a\cos\phi\frac{d\theta}{dt};$$

therefore
$$a^2(\tfrac{1}{2}m+m')\frac{d\theta}{dt}+m'r^2\frac{d\phi}{dt}=lm'v,$$

$$a^2(\tfrac{1}{2}m+m')\left(\frac{d\theta}{dt}\right)^2+m'r^2\left(\frac{d\phi}{dt}\right)^2=m'v^2.$$

The string will cease to coil round the pulley when $\frac{d\theta}{dt}=\frac{d\phi}{dt}$, and this condition gives
$$r'^2=l^2-a^2\left(\tfrac{1}{2}\frac{m}{m'}+1\right).$$

Let $h^2=a^2\left(\tfrac{1}{2}\frac{m}{m'}+1\right)$, $R^2=r^2+h^2-l^2$, and we have

$$\frac{1}{a}\frac{dr}{d\phi}=-\frac{lrR-hR^2}{h(l^2-h^2)},\ \text{whence}\ a\phi=2h\left\{\tan^{-1}\frac{r-R}{l-h}-\frac{\pi}{4}\right\}.$$

Also,
$$a\theta=2h\left\{\tan^{-1}\frac{r-R}{l-h}-\frac{\pi}{4}\right\}-l+r.$$

Again, $\quad \frac{d\phi}{dt}=\frac{v}{r}\cdot\frac{lr+hR}{h^2+r^2},\quad$ therefore $\quad \frac{1}{a}\frac{dr}{dt}=-\frac{v}{h}\cdot\frac{R}{r};$

and $at=\frac{h}{v}(h-R)$; hence, when the motion becomes uniform,

$$a\phi'=2h\left\{\tan^{-1}\left(\frac{l+h}{l-h}\right)^{\frac{1}{2}}-\frac{\pi}{4}\right\},$$

$$a\theta=2h\left\{\tan^{-1}\left(\frac{l+h}{l-h}\right)^{\frac{1}{2}}-\frac{\pi}{4}\right\}-l+(l^2-h^2)^{\frac{1}{2}},\quad at'=\frac{h^2}{v};$$

At this epoch $\frac{d\theta}{dt} = \frac{d\phi}{dt} = \frac{v}{l}$; and, if V be the velocity of P,

$$V^2 = \frac{v^2}{l^2}\left(l^2 - \tfrac{1}{2}a^2 \frac{m}{m'}\right).$$

Let $\qquad x' = a\cos\phi' - r'\sin\phi', \quad y' = a\sin\phi' + r'\cos\phi',$

$$\mu = \left(\frac{dy}{dx}\right) = \frac{r'\sin\phi' - a\cos\phi'}{r'\cos\phi' + a\sin\phi'} = -\frac{x'}{y'};$$

then the path of P is the straight line

$$y - y' = \mu(x - x'), \text{ or } x'x + y'y = a^2 + r'^2,$$

that is, perpendicular to CP.

7142. (By Capt. P. A. MacMahon.)—Deal n^n cards into n packs, one card from the top to each pack in succession. Let a second person choose a card, and name the pack in which it is. Gather up the cards so that the named pack may be the m_1th pack from the top. Repeat the above, placing the named pack m_2th from the top, and so on n times in succession. Find $m_1, m_2, m_3 \ldots m_n$, so that the card chosen (which is unknown to the dealer) may be the xth card from the top at the end of the nth deal, x being any number from 1 to n^n inclusive.

I. *Solution by the* Proposer; J. S. Jenkins, M.A.; *and others.*

Put $\qquad x + n^{n-1} + n^{n-2} + \ldots + n^2 + n = Q_1 n + R_1,$

$$Q_1 = Q_2 n + R_2, \quad Q_2 = Q_3 n + R_3, \quad \ldots\ldots, \quad Q_{n-1} = 0 \,.\, n + R_m,$$

where $R_1, R_2 \ldots$ may be numbers from 1 to n inclusive, but not zero.

Then will $\qquad m_1, m_2, \ldots m_n = R_1, R_2, \ldots R_n$ respectively.

II. *Solution by the* Rev. T. R. Terry, M.A.; Christine Ladd, B.A.; *and others.*

As the directions about dealing are not sufficiently clear, we may suppose that, when the cards are being dealt, they are held face downwards, but that, on being dealt onto the heaps, each card is turned face upwards. Thus, when preparing for a new deal, the dealer turns each heap face downwards, placing the named heap in the particular position assigned by the law. Required to find this law.

If at any stage of the proceedings the selected card is in the $(qn+r)$th place in its heap, and if this heap is then placed pth from the top, it is clear that after the next deal the card will be in the $[(p-1)\,n^{n-2} + (q+1)]$th place in the heap into which it has been dealt. Hence it follows at once that

$$x - 1 = (m_1 - 1) + (m_2 - 1)\,n + \ldots\ldots + (m_n - 1)\,n^{n-1}.$$

Thus the simple rule is: Express $(x-1)$ in the scale of n, let $r_1, r_2 \ldots r_n$ be the digits found in order, then $m_p = r_p + 1$, where p is any integer not greater than n.

6619. (By W. J. C. SHARP, M.A.)—With the notation adopted in Question 5493, prove that

$$\frac{(p^{i+1}-1)(p^{i+2}-1)\dots(p^{i+j}-1)}{(p-1)(p^2-1)\dots(p^j-1)} = \frac{p^{ij+1}-1}{p^j-1}\chi_{ij} + \frac{p^{ij}-1}{p-1}\chi_{ij-1}$$

$$+p\frac{p^{ij-1}-1}{p-1}\chi_{ij-2} + p\frac{p^{ij-2}-1}{p-1}\chi_{ij-3} + p^2\frac{p^{ij-3}-1}{p-1}\chi_{ij-4} + \&c.$$

[The Note on Question 5493, from the words "of course, when ij is even," applies to this Question as well.]

Solution by the PROPOSER.

If $\dfrac{(1-x^{i+1})(1-x^{i+2})\dots(1-x^{i+j})}{(1-x^2)(1-x^3)\dots(1-x^j)} \equiv B_0 + B_1 x + B_2 x^2 + \dots + B_{ij+1}x^{ij-1}$,

then, since $B_{ij+1-r} = -B_r$, we have

$$\frac{(1-x^{i+1})(1-x^{i+2})\dots(1-x^{i+j})}{(1-x)(1-x^2)\dots(1-x^j)}$$

$$\equiv B_0\frac{1-x^{ij+1}}{1-x} + B_1 x\frac{1-x^{ij-1}}{1-x} + B_2 x^2\frac{1-x^{ij-3}}{1-x} + \&c.$$

But B_r is the number of covariants of order j in the coefficients and weight of source possessed by a binary -ic, and $B_r = \chi_{ij-2r}$ (SALMON's *Higher Algebra*, pp. 118 and 272);

therefore
$$\frac{(x^{i+1}-1)(x^{i+2}-1)\dots(x^{i+j}-1)}{(x-1)(x^2-1)\dots(x^j-1)}$$

$$= \chi_{ij}\frac{x^{ij+1}-1}{x-1} + \chi_{ij-2}x\frac{x^{ij-1}-1}{x-1} + \chi_{ij-4}x^2\frac{x^{ij-3}-1}{x-1} + \&c.,$$

which (the superfluous terms vanishing of themselves) gives the correct result

$$\frac{(p^{i+1}-1)(p^{i+2}-1)\dots(p^{i+j}-1)}{(p-1)(p^2-1)\dots(p^j-1)} = \chi_{ij}\frac{p^{ij+1}-1}{p-1} + \chi_{ij-2}\,p\,\frac{p^{ij-1}-1}{p-1} + \&c.,$$

for all values of p.

It is evident that χ_{ij} is always unity, and the other χ's can always be determined by giving a sufficient number of values to p.

[Professor SYLVESTER's result in 5493 follows by making $p=1$, and adding the results when ij is even and when it is odd.]

7118. (By D. EDWARDES.)—Prove that (1) the area of the triangle formed by joining the feet of the normals drawn from a point (X, Y) to the parabola $y^2 = 4ax$, is $\Delta = a^{\frac{1}{2}}[4(X-2a)^3-27aY^2]^{\frac{1}{2}}$; (2) if the triangle is right-angled, $\Delta = a^{\frac{1}{2}}(4X^3-24aX^2+21a^2X+49a^3)^{\frac{1}{2}}$, and the hypotenuse cuts the axis at a distance from the vertex equal to one-fourth of the *latus rectum*.

Solution by J. W. SHARPE, M.A.; R. KNOWLES, B.A.; *and others.*

1. Write the equation to the parabola under the form $x = a\mu^2$, $y = 2a\mu$; then $a\mu^3 - (X - 2a)\,\mu - Y = 0$ gives the relation between the coordinates X, Y, and the parameters of the feet of the three normals from X, Y; and if these parameters be denoted by μ, μ_1, μ_2, we have

$$\Delta = a^2(\mu_1 - \mu_2)(\mu_2 - \mu)(\mu - \mu_1).$$

Now, if we form the equation whose roots are the squares of the differences between the roots of the equation $x^3 + qx + r = 0$, its absolute term is found to be $27r^2 + 4q^3$; whence

$$a^2(\mu_1 - \mu_2)(\mu_2 - \mu)(\mu - \mu_1) = [4(X - 2a)^3 - 27aY^2]^{\frac{1}{2}}.$$

2. In the second case, we may put $\mu_1\mu_2 = -1$; and then the chord joining the points μ_1 and μ_2, which is the hypotenuse, is

$$2x - (\mu_1 + \mu_2)\,y + 2a\mu_1\mu_2 = 0,$$

which cuts the axis at the point $x = a$, that is, at a distance from the vertex equal to a quarter of the *latus rectum*. Also, in this case, we have

$$\mu\mu_1\mu_2 \text{ or } -\mu = \frac{Y}{a}, \quad \mu + \mu_1 + \mu_2 = 0, \quad \mu_1\mu_2 + \mu(\mu_1 + \mu_2) = -\frac{X - 2a}{a};$$

therefore $\qquad 1 + \mu^2 = \dfrac{X - 2a}{a}$, therefore $\dfrac{Y^2}{a^2} = \dfrac{X - 3a}{a}$;

therefore $\qquad \Delta = a^{\frac{1}{2}}[4X^3 - 24aX^2 + 21a^2X + 49a^3]^{\frac{1}{2}}.$

7061 & 7095. (By J. HAMMOND, M.A.)—Prove that, (1) if we have

$$\phi(x) = \phi\left(\frac{c}{x-1}\right), \quad \text{then} \int_0^\infty \frac{\phi(x)\,dx}{x^2 - x - c} = \tfrac{1}{2}\int_0^1 \frac{\phi(x)\,dx}{x^2 - x - c};$$

and (2), if we have $\qquad \phi(x) = \phi\left(\sin^2\alpha + \dfrac{\cos^2\alpha}{x}\right),$

then $$\int_0^\infty \frac{\phi(x)\,dx}{(x-1)(x+\cos^2\alpha)} = 2\int_0^1 \frac{\phi(x)\,dx}{(x-1)(x+\cos^2\alpha)} = 0,$$

and also $$\int_0^\infty \frac{\phi(x)\,dx}{(x-1)^2} = (1 + \sec^2\alpha)\int_0^1 \frac{\phi(x)\,dx}{(x-1)^2}.$$

Solution by D. EDWARDES; J. O'REGAN; *and others.*

1. $$\int_0^\infty \frac{\phi(x)}{x^2 - x - c}\,dx = \int_\infty^1 \frac{\phi(y)}{y^2 - y - c}\,dy \quad \left(\text{if } x = \frac{c}{y-1}\right)$$

$$= \int \frac{\phi(x)}{x^2 - x - c}\,dx - \int_0^\infty \frac{\phi(x)}{x^2 - x - c}\,dx,$$

$$2\int_0^\infty \frac{\phi(x)}{x^2 - x - c}\,dx = \int_0^1 \frac{\phi(x)}{x^2 - x - c}\,dx.$$

2. $$\int_0^1 \frac{\phi\left(\sin^2\alpha + \dfrac{\cos^2\alpha}{x}\right)}{(x-1)(x+\cos^2\alpha)}\,dx = \int_\infty^1 \frac{\phi(y)}{(y-1)(y+\cos^2\alpha)}\,dy,$$
$$\left(\text{if } x = \frac{\cos^2\alpha}{y-\sin^2\alpha}\right),$$

therefore $$\int_1^\infty \frac{\phi(x)}{(x-1)(x+\cos^2\alpha)}\,dx + \int_0^1 \frac{\phi(x)}{(x-1)(x+\cos^2\alpha)}\,dx = 0,$$

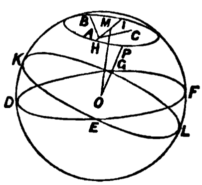
therefore $$\int_0^\infty \frac{\phi(x)}{(x-1)(x+\cos^2\alpha)}\,dx = 0.$$

Also $$\int_0^1 \frac{\phi\left(\sin^2\alpha + \dfrac{\cos^2\alpha}{x}\right)}{(x-1)^2}\,dx = \cos^2\alpha \int_1^\infty \frac{\phi(y)\,dy}{(y-1)^2} \left(\text{if } x = \frac{\cos^2\alpha}{y-\sin^2\alpha}\right),$$

or $$(1+\cos^2\alpha)\int_0^1 \frac{\phi(x)}{(x-1)^2}\,dx = \cos^2\alpha \int_0^\infty \frac{\phi(x)}{(x-1)^2}\,dx.$$

6250. (By Professor Seitz, M.A.)—A, B, C are random points within a hemisphere; find the chance that the plane through A, B, C does not intersect the base of the hemisphere.

Solution by the Proposer; Professor Nash, M.A.; *and others.*



Let DEFG be the base of the hemisphere, O its centre, OM the perpendicular on the plane through A, B, C, HAI a diameter of the section made by this plane, OP a line such that AB is parallel to the plane MOP, and KELG a great circle of the complete sphere, perpendicular to OP.

Now, instead of confining the points A, B, C to the given hemisphere, we may confine them to the hemisphere whose base is KELG, and find the chance that the plane through them does not intersect the base KELG. The result will evidently be the same as that required in the question. Let OH $= r$, MA $= x$, AB $= y$, AC $= z$, $\angle$ HOM $= \theta$, $\angle$ BAM $= \phi$, $\angle$ CAM $= \psi$, $\angle$ MOP $= \mu$, and the angle the plane MOP makes with a fixed plane through OP $= \omega$.

Then an element of the sphere at A is $r\sin\theta\,d\theta\,.\,2\pi x\,dx$, at B it is $y^2\,dy\,d\phi\,d\mu$, and at C it is $\sin(\phi+\psi)\sin\mu\,z^2\,dz\,d\psi\,d\omega$. The limits of θ are 0 and $\frac{1}{2}\pi$; of x, 0 and $2r\sin\theta = x'$, and tripled; of ϕ, $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; of ψ, $-\phi$ and $\frac{1}{2}\pi$, and doubled; of μ, 0 and $\frac{1}{2}\pi - \theta$; of ω, 0 and 2π; of y, 0 and $2x\cos\phi = y'$; and of z, 0 and $2x\cos\psi = z'$. Hence, since the whole number of ways the three points can be taken within a hemisphere is $(\frac{2}{3}\pi r^3)^3$, the required chance is

$$\frac{6}{(\frac{2}{3}\pi r^3)^3}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\phi}^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\theta}\int_0^{2\pi}\int_0^{y'}\int_0^{z'} r\sin\theta\,d\theta\,.\,2\pi x\,dx\,d\phi\sin(\phi+\psi)\,d\psi \times \sin\mu\,d\mu\,d\omega\,y^2\,dy\,z^2\,dz$$

$$= \frac{108}{\pi^2 r^8} \int_0^{\frac12\pi} \int_0^{x'} \int_{-\frac12\pi}^{\frac12\pi} \int_{-\phi}^{\frac12\pi'} \int_0^{\frac12\pi-\theta} \int_0^{2\pi} \int_0^{y'} \sin\theta\, d\theta\, x^4\, dx\, d\phi \cos^3\psi \sin(\phi+\psi)\, d\psi \times \sin\mu\, d\mu\, d\omega\, y^2\, dy$$

$$= \frac{288}{\pi^2 r^8} \int_0^{\frac12\pi} \int_0^{x} \int_{-\frac12\pi}^{\frac12\pi} \int_{-\phi}^{\frac12\pi} \int_0^{\frac12\pi-\theta} \int_0^{2\pi} \sin\theta\, d\theta\, x^7\, dx \cos^3\phi\, d\phi \cos^3\psi \sin(\phi+\psi)\, d\psi \times \sin\mu\, d\mu\, d\omega$$

$$= \frac{576}{\pi r^8} \int_0^{\frac12\pi} \int_0^{x'} \int_{-\frac12\pi}^{\frac12\pi} \int_{-\phi}^{\frac12\pi} \int_0^{\frac12\pi-\theta} \sin\theta\, d\theta\, x^7\, dx \cos^3\phi\, d\phi \cos^3\psi \sin(\phi+\psi)\, d\psi \times \sin\mu\, d\mu$$

$$= \frac{576}{\pi r^8} \int_0^{\frac12\pi} \int_0^{x'} \int_{-\frac12\pi}^{\frac12\pi} \int_{-\phi}^{\frac12\pi} (1-\sin\theta) \sin\theta\, d\theta\, x^7\, dx \cos^3\phi\, d\phi \cos^3\psi \sin(\phi+\psi)\, d\psi$$

$$= \frac{72}{\pi r^8} \int_0^{\frac12\pi} \int_0^{x'} \int_{\frac12-\pi}^{\frac12\pi} \left[3\left(\tfrac12\pi+\phi\right)\sin\phi + 2\cos\phi + \sin^2\phi\cos\phi\right] (1-\sin\theta)\sin\theta\, d\theta \times x^7\, dx \cos^3\phi\, d\phi$$

$$= \frac{315}{4r^8} \int_0^{\frac12\pi} \int_0^{x'} (1-\sin\theta)\sin\theta\, d\theta\, x^7\, dx = \frac{315}{32}\int_0^{\frac12\pi} (1-\sin\theta)\sin^9\theta\, d\theta = 4 - 5\pi\left(\frac{63}{128}\right)^2.$$

7041. (By W. J. C. Sharp, M.A.)—Find the form of the catenary in which an extensible string, uniform when unstretched, will hang on the centre of gravity.

Solution by the Proposer.

Generally, if s be the arc of any catenary from a fixed point to any point P, and θ the angle between the tangents at these points; and tds, nds be the tangential and normal forces on ds situated at P, and T the tension at that point;

$$\frac{dT}{ds} = t \quad\text{and}\quad T\frac{d\theta}{ds} = u;$$

and for gravitation catenaries, if the tangent at the initial point be horizontal, and ω be the weight of the string at P per unit of length;

$$= -\omega\sin\theta, \quad n = -\omega\cos\theta, \quad\text{and therefore}\quad T = -a\sec\theta,$$

and
$$\frac{ds}{d\theta} = \frac{a}{\omega}\sec^2\theta.$$

In the question, ω is a variable which is connected with ω_0, the weight of the unstretched string per unit of length, by the equation

$$\omega = \frac{\omega_0}{1+\dfrac{T}{\lambda}} = \frac{\omega_0}{1+\dfrac{a}{\lambda}\sec\theta};$$

therefore
$$\frac{ds}{d\theta} = \frac{a}{\omega_0}\left(1+\frac{a}{\lambda}\sec\theta\right)\sec^2\theta.$$

$$s = \frac{a}{\omega_0}\tan\theta + \frac{a^2}{2\omega_0\lambda}\sec\theta\tan\theta + \frac{a^2}{2\omega_0\lambda}\log\frac{1+\sin\theta}{\cos\theta} \quad \ldots\ldots\ldots\ldots(1),$$

no constant being added, since s and θ vanish together.

If s_0 is the unstretched length of s, we have

$$s_0 = \int_0^s \frac{ds}{1+\frac{a}{\lambda}\sec\theta} = \frac{a}{\omega_0}\tan\theta.$$

Equation (1) is the intrinsic equation to the catenary.

If (x, y) be the Cartesian coordinates of P, the origin being at the initial point and the axes horizontal and vertical,

$$x = \int_0^s \cos\theta\, ds = \frac{a}{\omega_0}\log\frac{1+\sin\theta}{\cos\theta} + \frac{a^2}{\omega_0\lambda}\tan\theta,$$

$$y = \int_0^s \sin\theta\, ds = \frac{a}{\omega_0}(\sec\theta - 1) + \frac{a^2}{2\omega_0\lambda}\tan^2\theta;$$

also, since $$p = \frac{a}{\omega_0}\left(1+\frac{a}{\lambda}\sec\theta\right)\sec^2\theta,$$

there is a cusp when $$\theta = \cos^{-1}\left(-\frac{a}{\lambda}\right).$$

7117. (By R. Knowles, B.A., L.C.P.)—Show that the greatest value of x for which $(1.2.3.4\ldots 2^n) \div 2^x$ is an integer, is 2^{n-1}.

Solution by Belle Easton; J. W. Sharpe, M.A.; *and others.*

The numbers not greater than 2^n, which contain powers of 2 as factors,

are $2^n,\ 2^{n-1},\ 2^{n-2}\times 1$ or $3,\ 2^{n-3}\times 1$ or 3 or 5, &c.,

and in general those which contain 2^{n-r} are formed by $2^{n-r}\times$ each and every odd number from 1 up to (2^r-1); but there are 2^{r-1} of these odd numbers; hence the power of 2 contained in the product of these 2^{r-1} numbers is $(n-r)\,2^{r-1}$; hence the maximum value of x is given by

$$x = n + n - 1 + 2(n-2) + 2^2(n-3) + \&c. + 2^{n-2};$$

$$\therefore\ 2x = 2n \qquad + 2(n-1) + 2^2(n-2) + \&c. + 2^{n-2}\,2 + 2^{n-1};$$

therefore, by subtraction, we have

$$x = 1 + 2 + 2^2 + \&c. + 2^{n-1} = 2_n - 1.$$

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